# Finite Elements  and Particle methods for Industrial Applications

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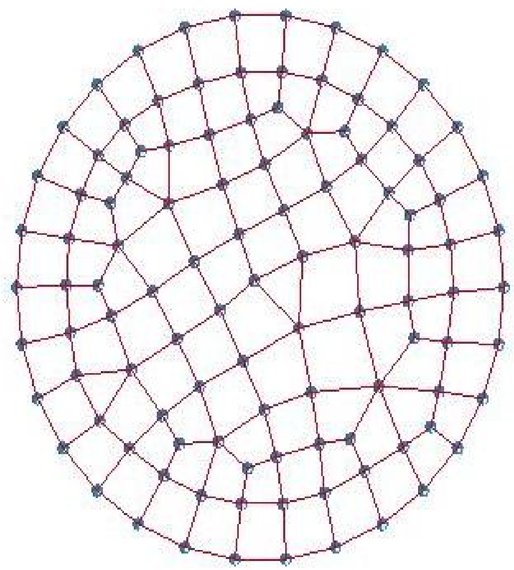
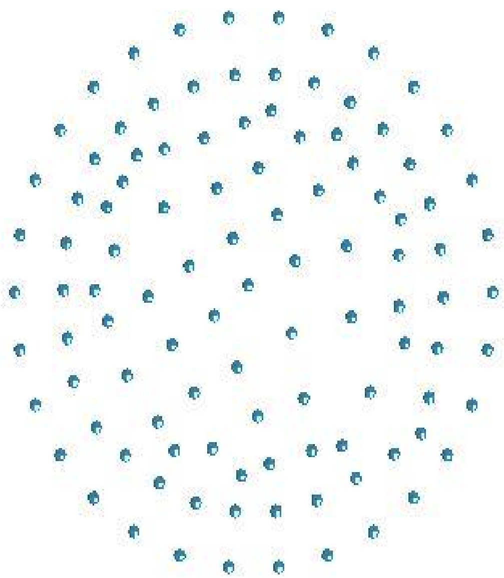
# Introducing FEM and SPH Methods





# Introducing SPH Method

1) Unlike Molecular dynamic Analysis , SPH method is a deterministic method and not a statistical method.

2) Corpuscular Method is a statistical method and solves for velocity Distribution, or the probability of having a specific velocity.

3) Corpuscular method solves for Maxwell-Bolzmann distribution equation for velocity

# Introducing SPH Method

1) Like FEM Method, SPH method uses conservation equations for continuum Mechanics to solve for velocity, pressure and energy.

$$\frac{d\rho}{dt} = -\rho . \nabla . \vec{v}$$

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho} . \nabla . \sigma + f_{ext}$$

$$\frac{de}{dt} = -\frac{1}{\rho} . \sigma . \nabla . \vec{v}$$

# Introducing SPH Method

1) In FEM Method a ***weak formulation*** is used to solve Conservative equations

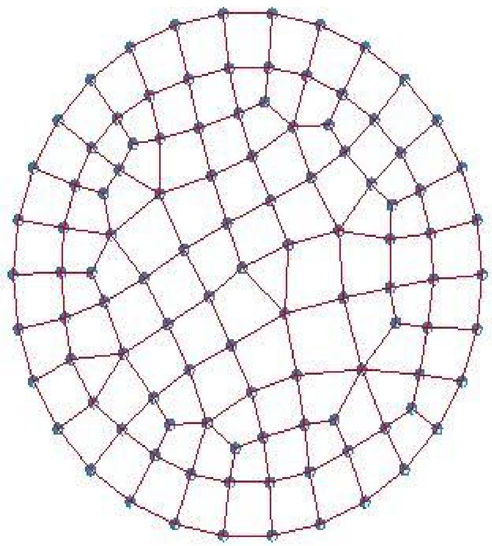
2) In SPH method we use a ***collocation method*** . to solve Conservative equations

$$\frac{d\rho}{dt} = -\rho . \nabla . \vec{v}$$

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho} . \nabla . \sigma + f_{ext}$$

$$\frac{de}{dt} = -\frac{1}{\rho} . \sigma . \nabla . \vec{v}$$

# Lagrangian FEM  and SPH Formulations



Cylindrical mesh and nodes

# Why do we need the mesh ?

Unlike FEM Method, because of the missing mesh the SPH method suffers from:

1) Function interpolation

2) Support domain different from Influence Domain

3) Lack of Consistency

4) Tensile Instability

5) Boundary Conditions

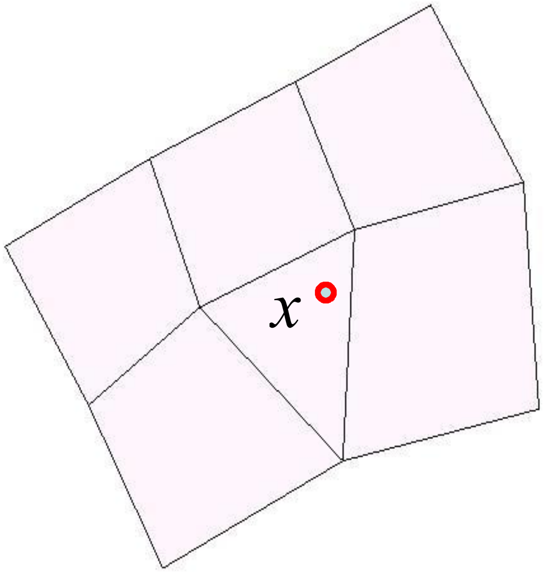
***Question:***

***How to remedy to these problems in SPH ?***

# Function interpolation

In FEM we need the mesh for:

1) Function Interpolation at any location. $x$




$$u(x) = \sum_j u_j . N_j(x)$$

2) Derivative of Function at any location.

$$\nabla u(x) = \sum_j u_j \nabla . N_j(x)$$

$N_j(x)$ Shape function at node j

# Function interpolation

In SPH method, we need to define:

1) Interpolation Function

2) Derivation of function, to solve conservative equations

$$\frac{d\rho}{dt} = -\rho . \nabla . \vec{v}$$

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho} . \nabla . \sigma$$

$$\frac{de}{dt} = -\frac{1}{\rho} . \sigma . \nabla . \vec{v}$$

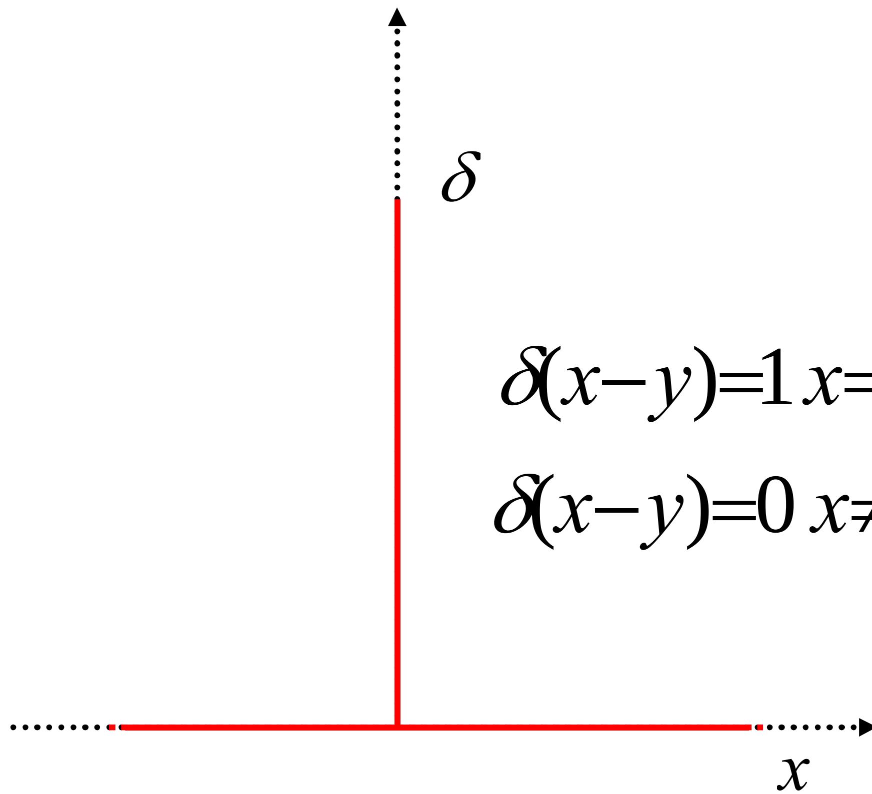
# Integral interpolation

At any location $x$ the integral interpolation of the function $u(x)$ is defined:

$$u(x)=\int_{\Omega} u(y).\delta(x-y).dy$$

$\delta$ : DIRAC function satisfies:

$$\int_{\Omega} \delta(x-y).dy=1$$




$$\delta(x-y)=1 \quad x=y$$

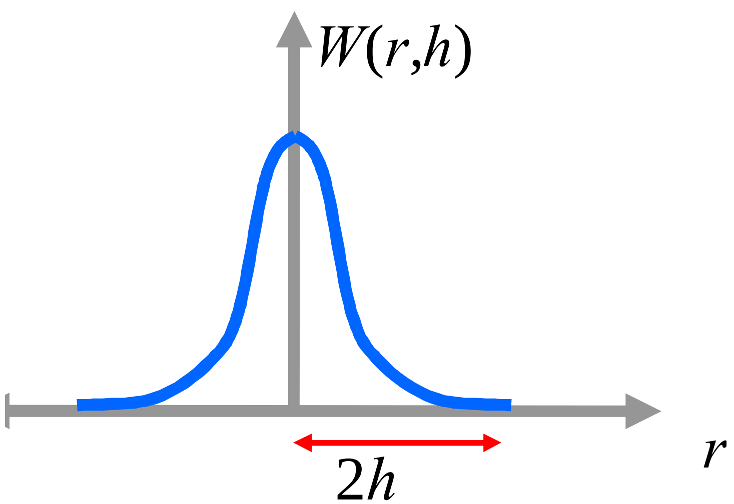
$$\delta(x-y)=0 \quad x\neq y$$

# Integral interpolation

The Dirac Function is approached by the Kernel Function $W(r,h)$

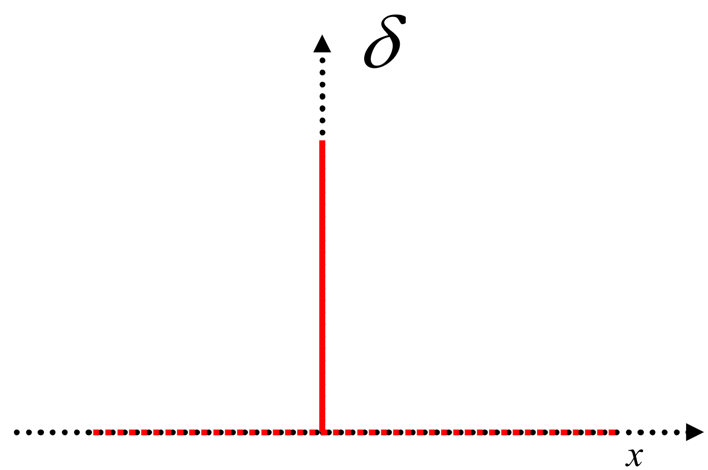
$$\int_{\Omega} W(r,h)\,dr = 1$$

$$h \rightarrow o \quad \Rightarrow \quad W(r,h) \rightarrow \delta_r$$




$\longrightarrow$

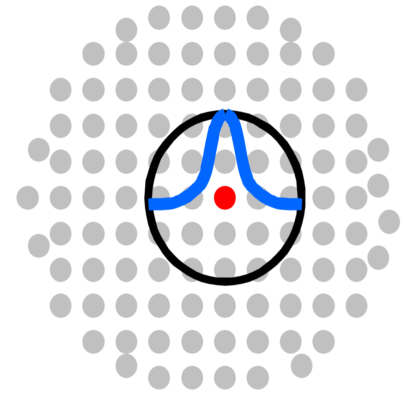
$h \rightarrow 0$



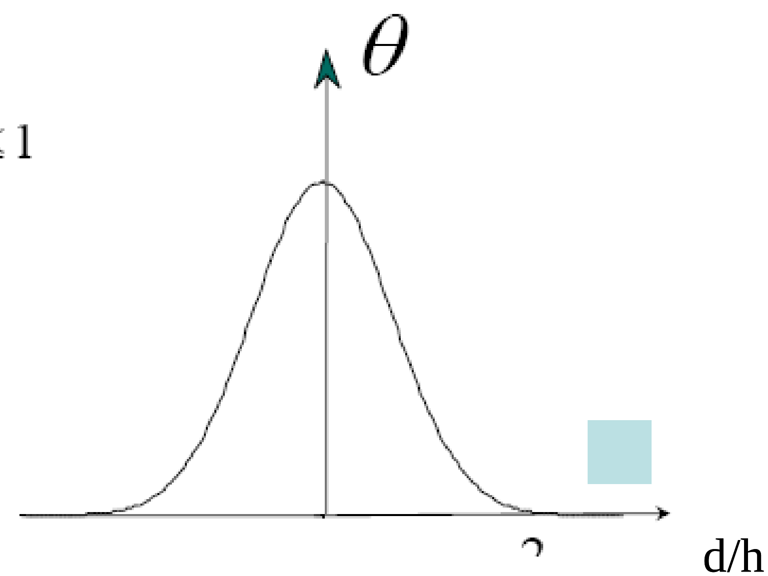
# Integral interpolation

The Kernel Function W is defined by:

$$W(d,h) = \frac{1}{h^{\alpha}} . \theta\left(\frac{d}{h}\right)$$

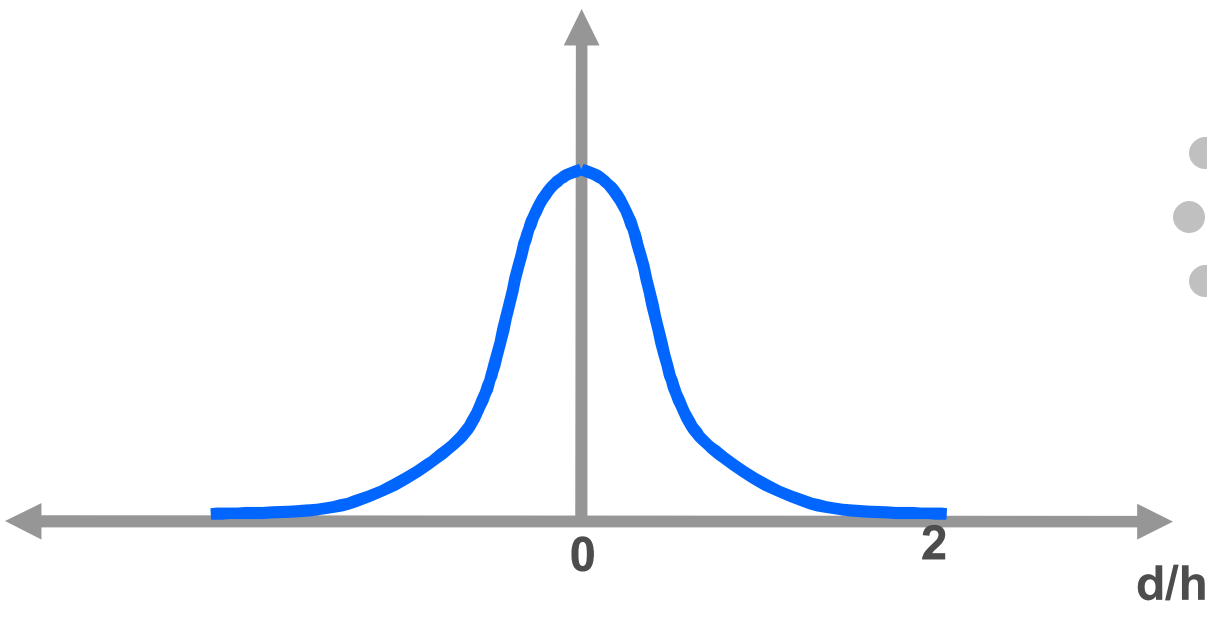


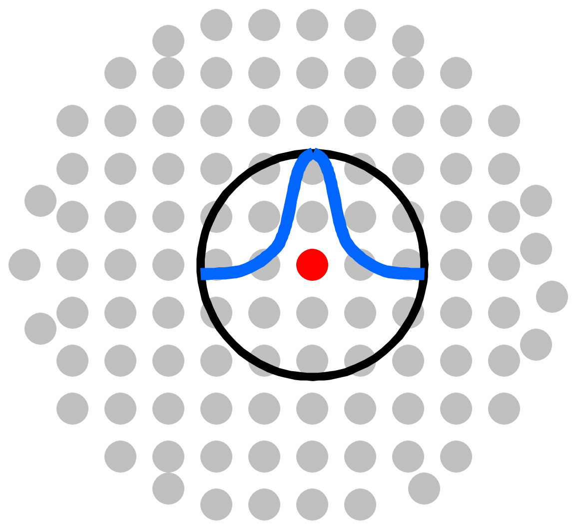
$$\theta(d) = C \times \begin{cases} 1 - \frac{3}{2}d^2 + \frac{3}{4}d^3 & \text{si} \quad 0 \leq /d/ \leq 1 \\ \frac{1}{4}(2-d)^3 & si \quad 1 \leq /d/ \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$



# Integral interpolation

Kernel function for 2D problem



# Interpolation Consistency

A central issue in SPH method is how to perform function interpolation with consistency with no mesh

Unlike FEM, SPH method cannot reproduce:

1) Constant function

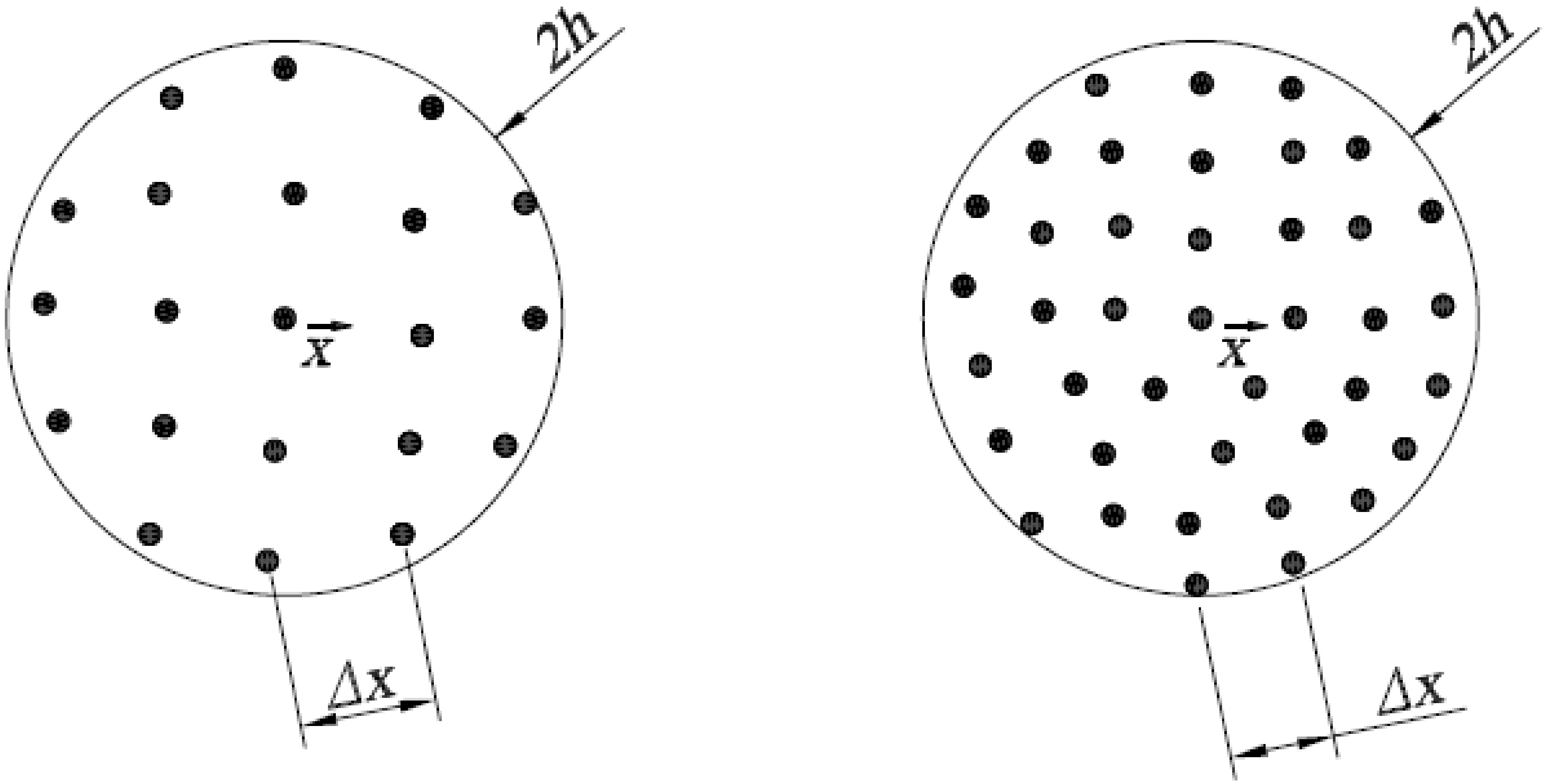
u(x)=1 $$\sum_j N_j(x)=1 \qquad \sum_j \omega_j . W(x - x'_j, h) \neq 1$$

2) Linear function

u(x)= x $$\sum_j x_j N_j(x)=x \qquad \sum_j \omega_j . x_j . W(x - x'_j, h) \neq x$$

Why do we need SPH to reproduce constant and linear function ??

# Smoothing length



# Consistency of constant function

u constant function: u(x)=1

$$\sum_j N_j(x) = 1 \qquad \sum_j \omega_j . W(x - x'_j, h) \neq 1$$

For constant function:

FEM Interpolation is exact

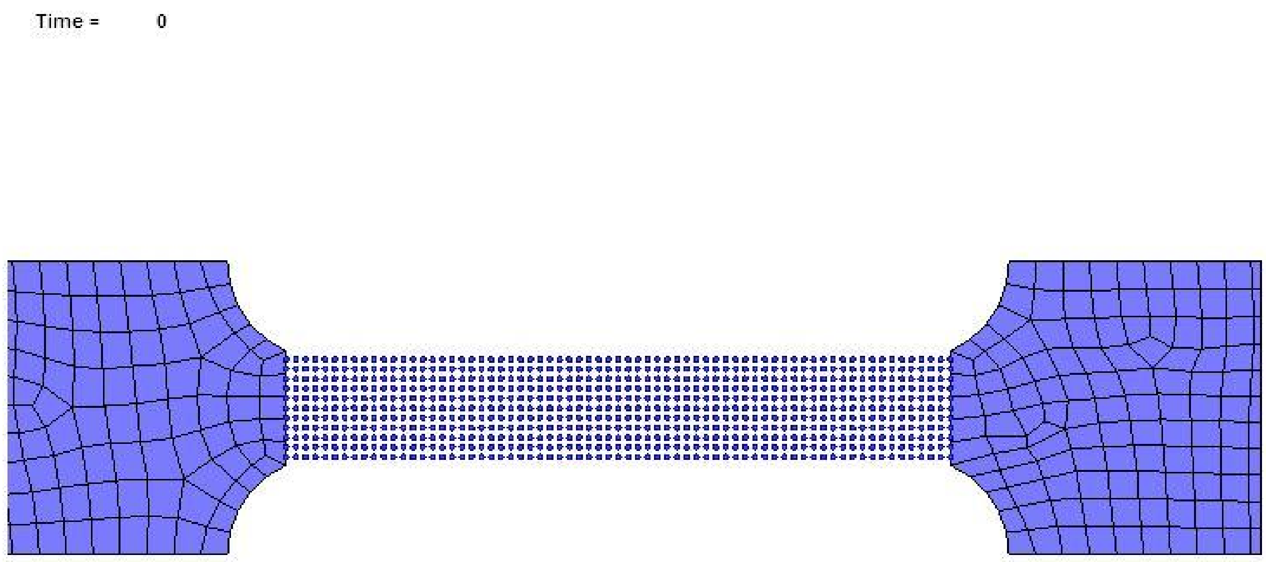
SPH Interpolation is not exact.

SPH Interpolation does not reproduce constant functions

# Tensile Instability

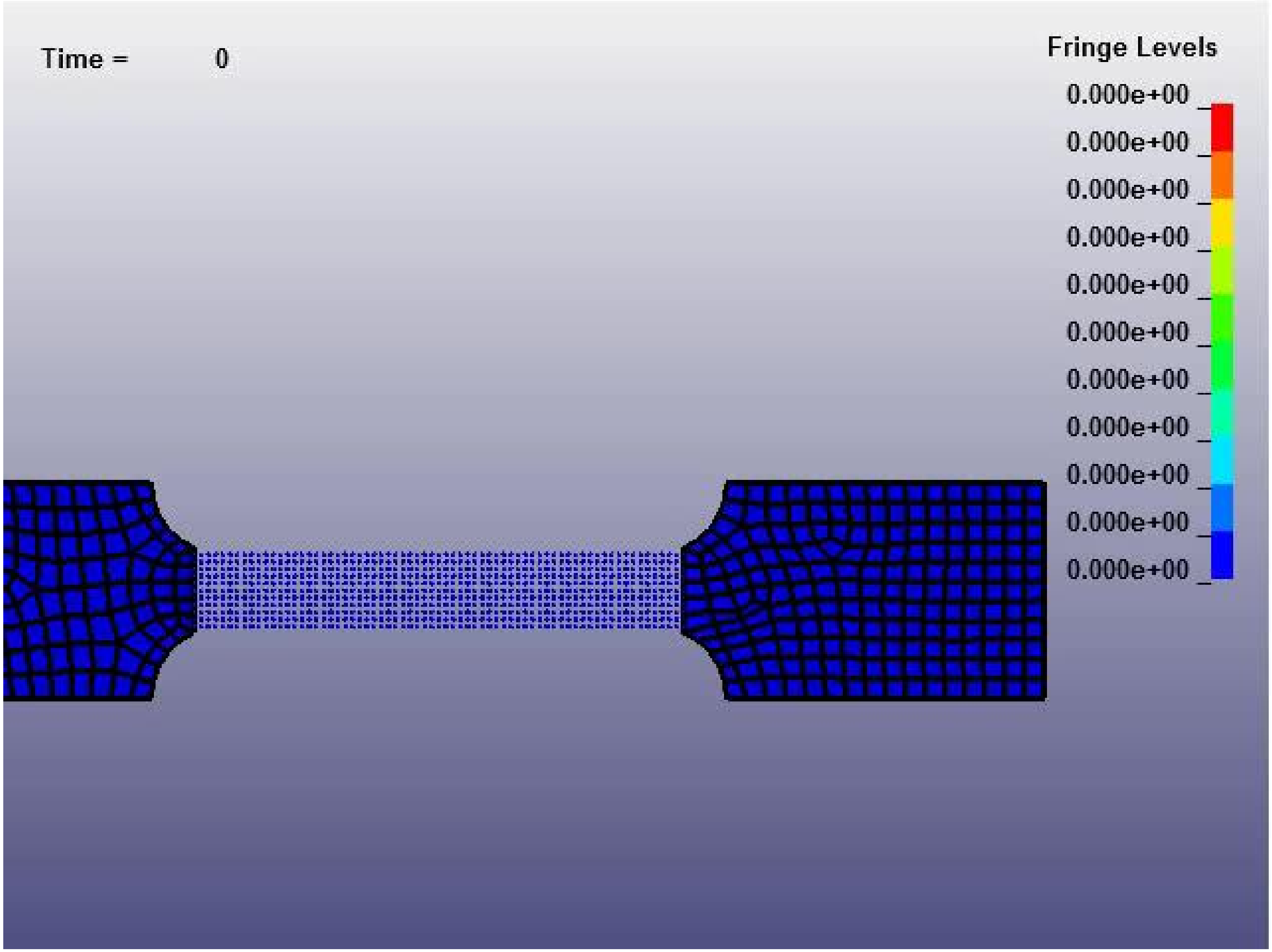
Tensile instability occurs when particles are under tensile stress.

The motion of the particles become unstable



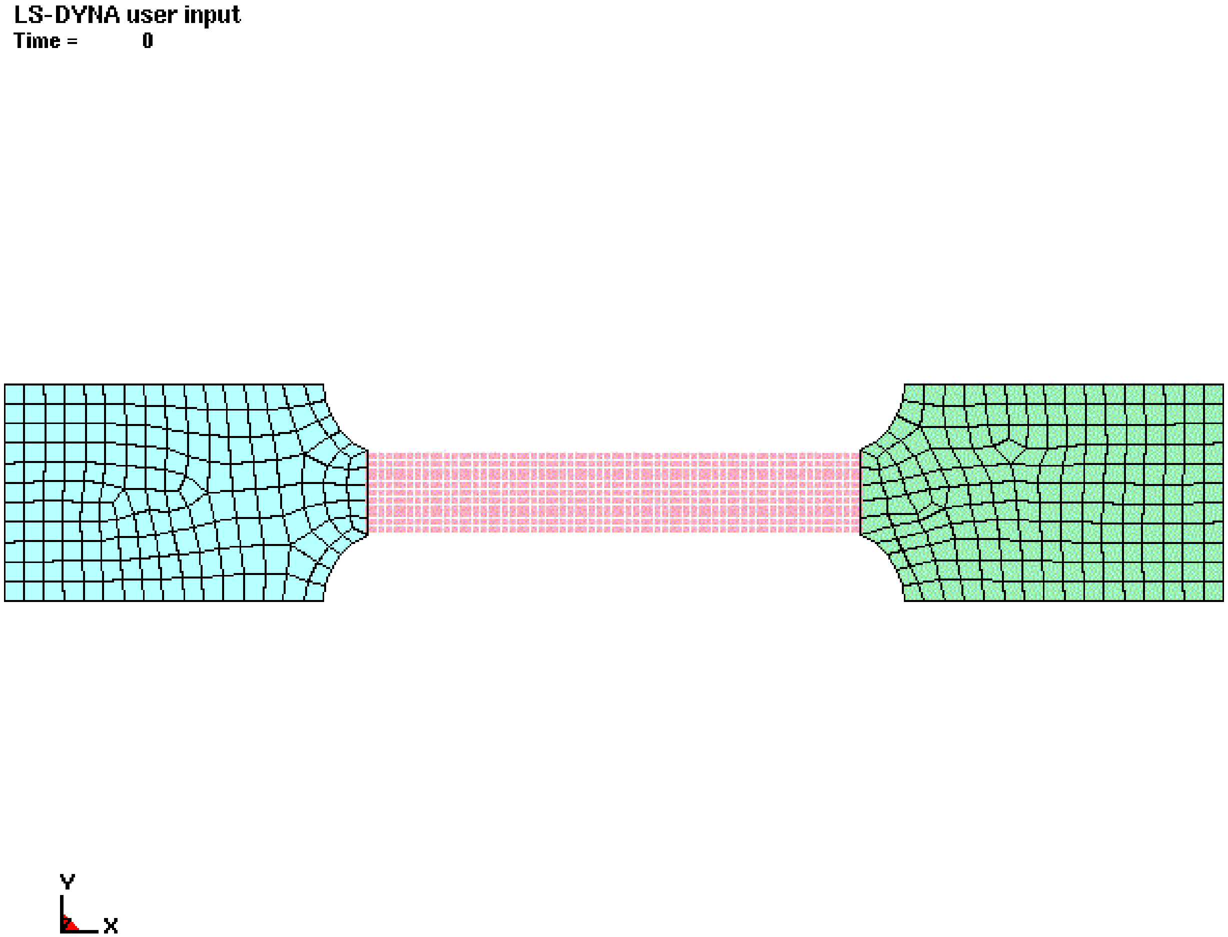
# Tensile Instability

Eulerian Kernel



# Tensile Instability

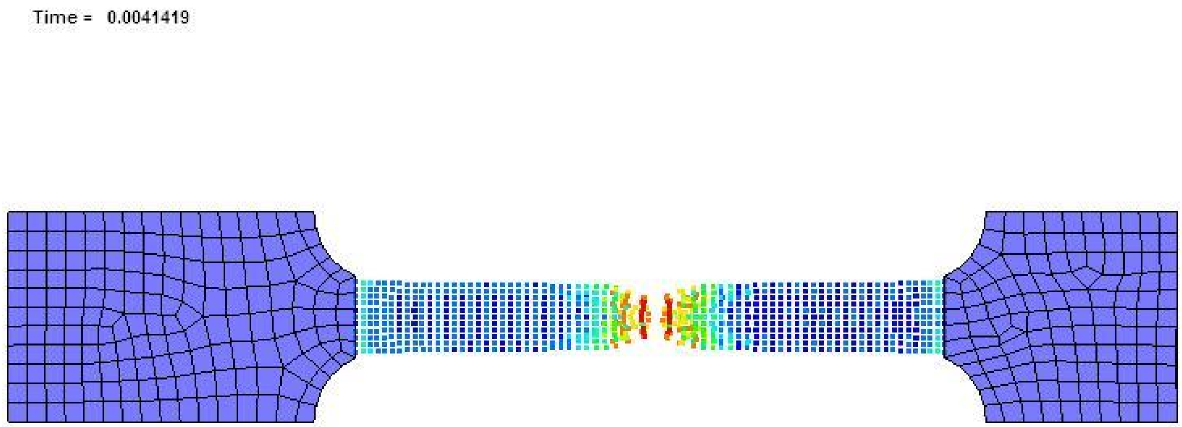
Lagrangian Kernel



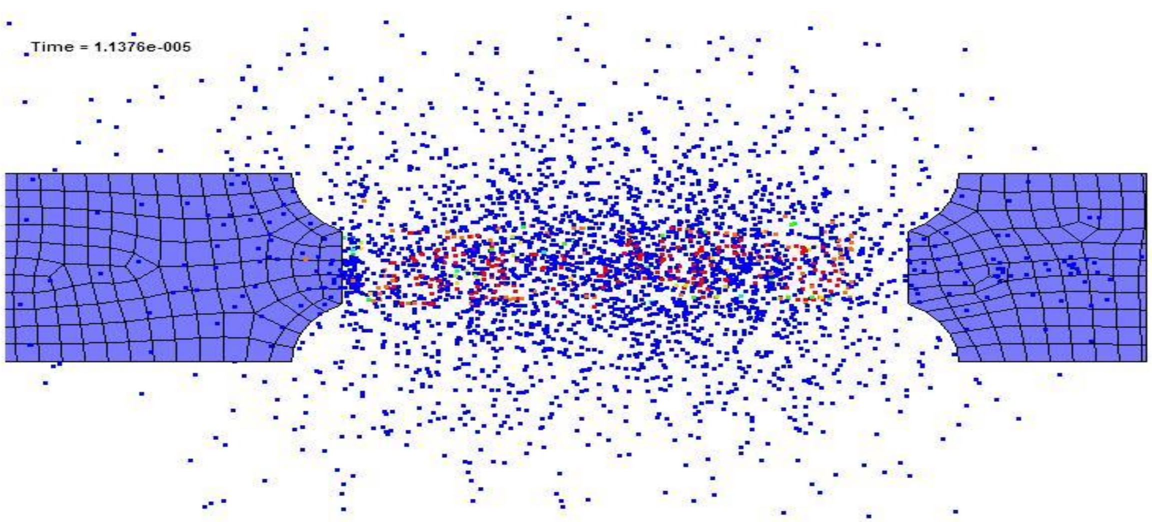

In the Lagrangian Kernel, the particle volume and the smoothing length are from initial configuration. The particle neighbors do not change with time

The Lagrangian Kernel is not suitable for problems of fluid flow

# Tensile Instability

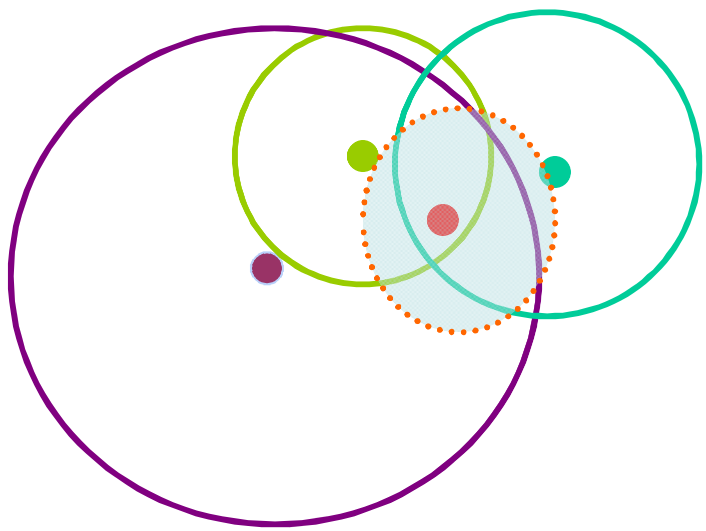



Lagrangian Kernel




Eulerian Kernel

## Support and Influence Domain in SPH



**Particle** ● **influences all Particle** ● ● ●

● ● ● are in the influence domain of ● and not in the support domain

Influence domain of Particle is different from support domain

# Boundary Conditions

$$u(x_i) = \int_{\Omega} u(y).W(x_i - y, h).dy \quad \longrightarrow \quad u(x_i) = \sum_j \omega_j .u_j .W(x_i - x_j .h)$$

$$u'(x) = \int_{\Omega} u'(y).W(x - y, h).dy \quad \longrightarrow \quad u'(x_i) = \sum_j \omega_j .u'_j .W(x_i - x_j .h)$$

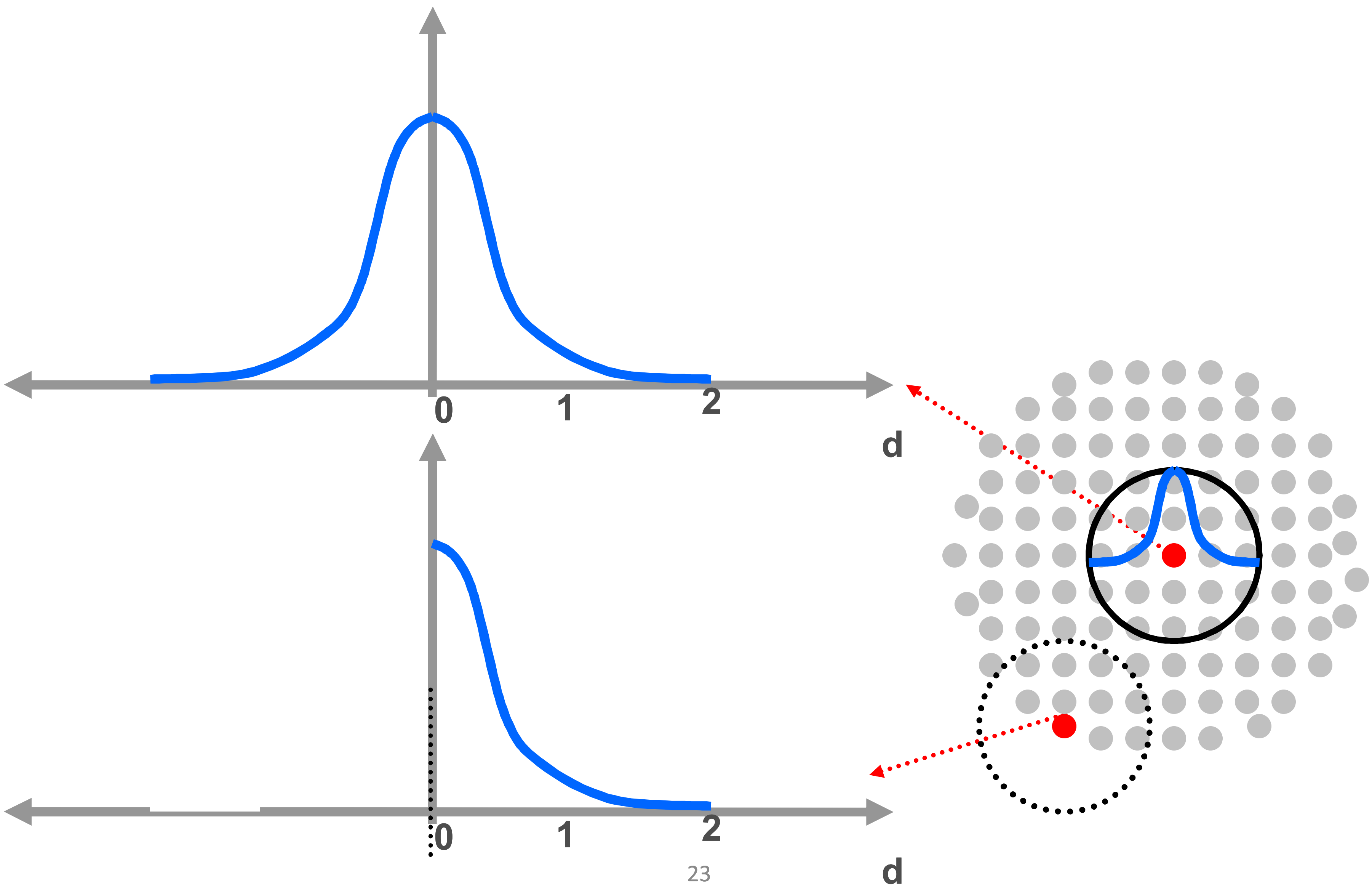
As in FEM we want to have $u'(x_i) = \sum_j \omega_j .u_j .W'(x_i - x_j .h)$

This is not true if the particle i is on the boundary

$$\int_{\Omega} u'(y).W(x - y, h).dy = -\int_{\Omega} u(y).W'(x - y, h).dy + \int_{boundary} u(y).W(x - y, h).dy$$

$\int_{boundary} u(y).W(x - y, h).dy \neq 0$ for particle near the boundary

# Boundary Conditions



## Approximation of conservation Laws

For each particle I, we solve:

$$\frac{d}{dt}\rho_i = -\rho_i \sum_j \frac{m_j}{\rho_j}(v_j - v_i)W_{ij}^{'}$$

$$\frac{d}{dt}v_i = \sum_j -m_j\left(\frac{\sigma_i}{\rho_i^2} + \frac{\sigma_j}{\rho_j^2}\right)W_{ij}^{'}$$

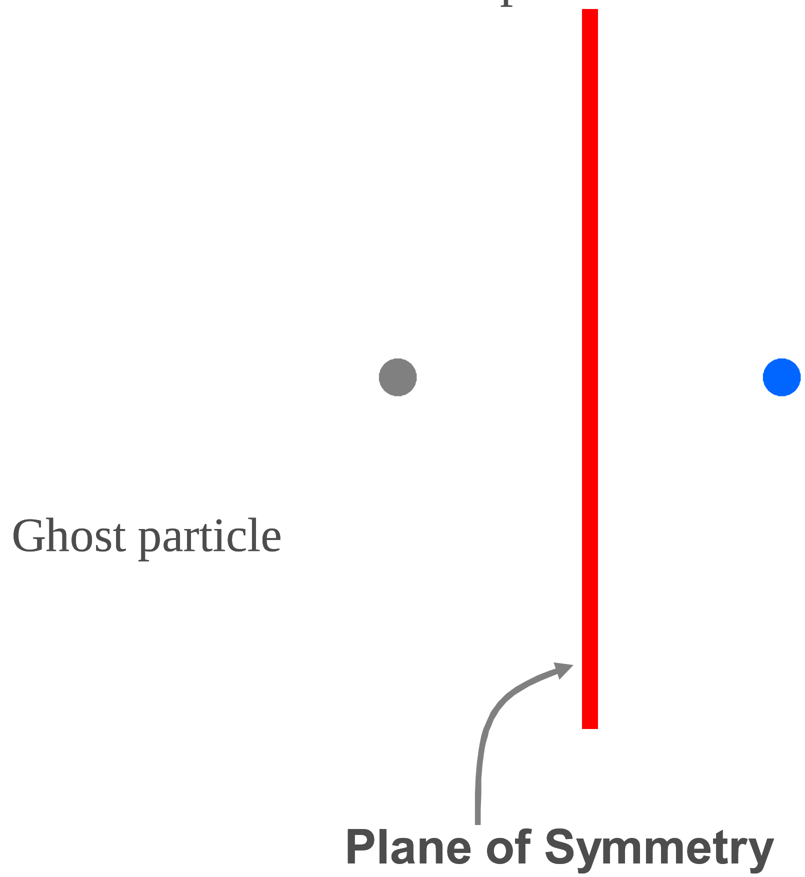
$$\frac{d}{dt}e_i = \frac{P_i}{\rho_i^2}\sum_j m_j(v_j - v_i)W_{ij}^{'}$$

No kernel function $W_{ij}$ involved in conservative equations

Only derivative of the kernel $W_{ij}^{'}$ involved

BOUNDARY_SPH_SYMMETRY_PLANE

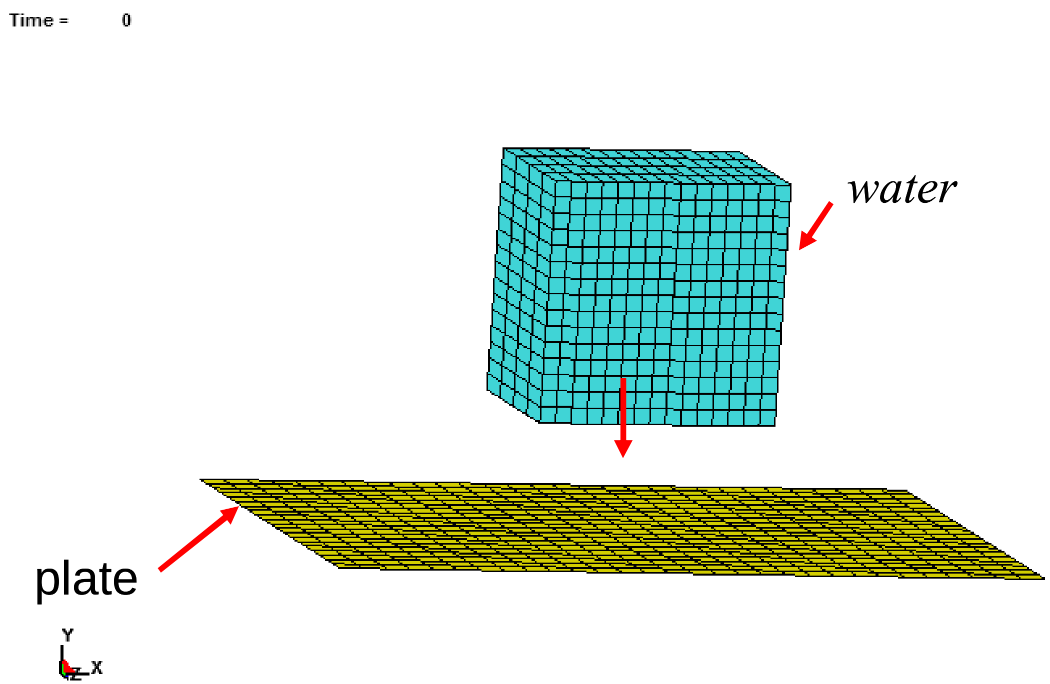
- Creates GHOST particles



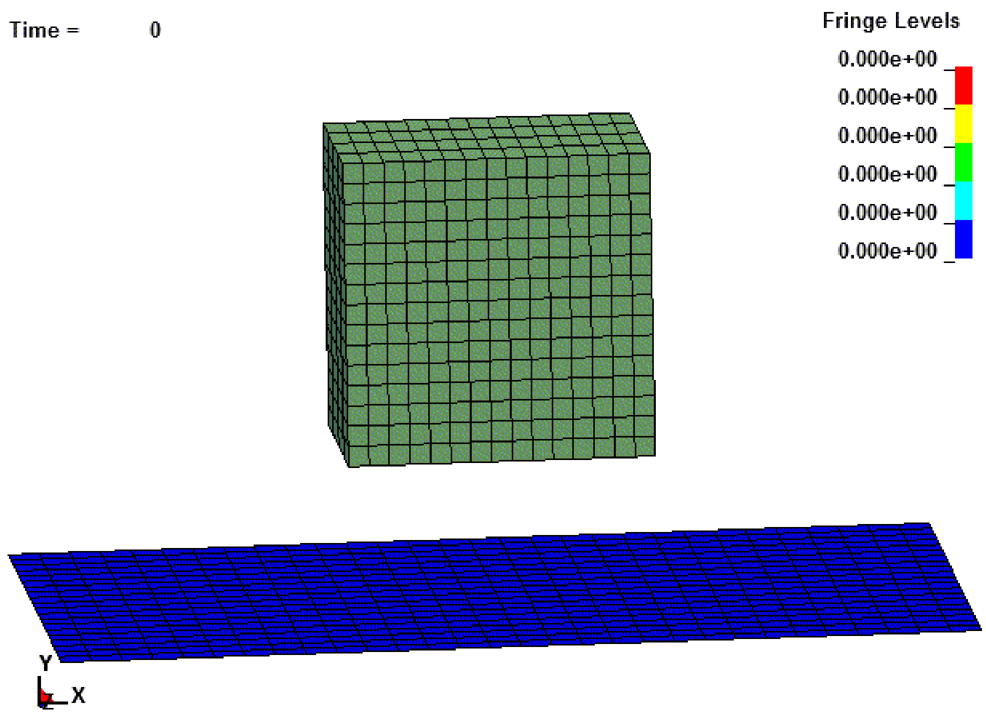
## Lagrangian and SPH meshes in 3D

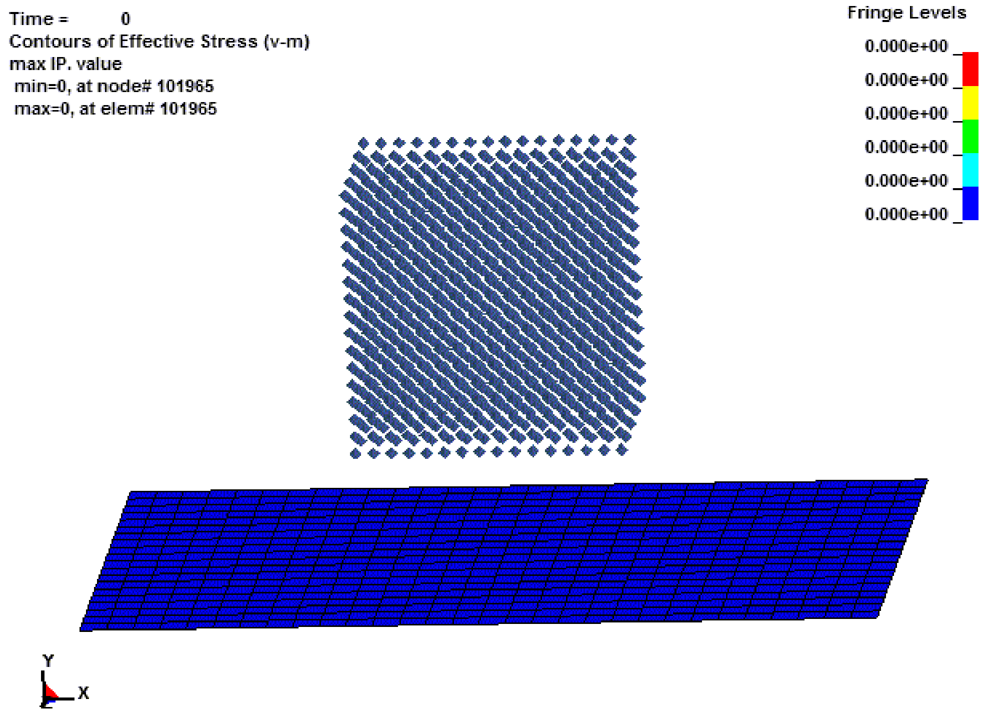
How the SPH mesh should be compared to the Lagrangian mesh
Same mesh or finer mesh ?

Water impacting a plate

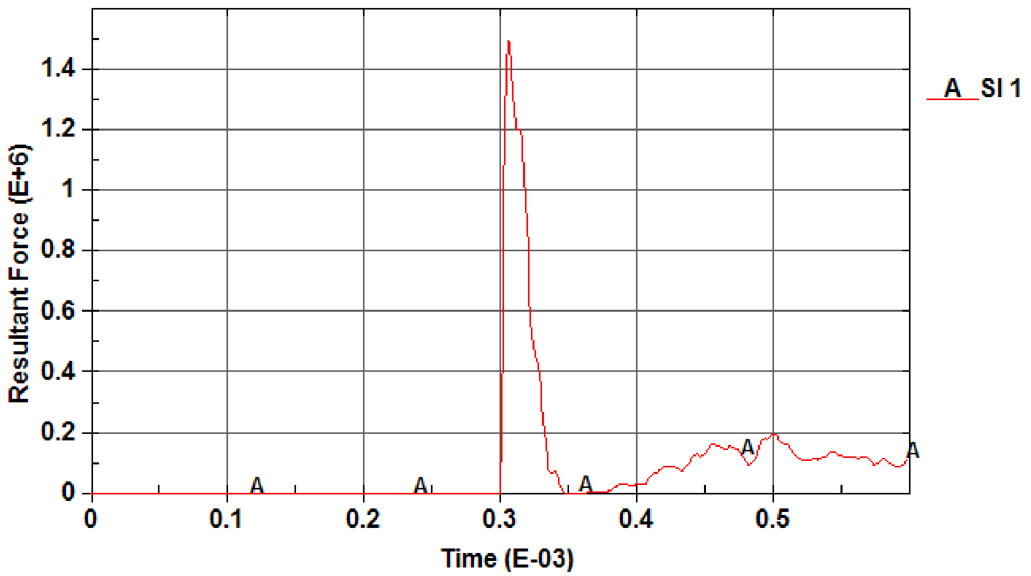


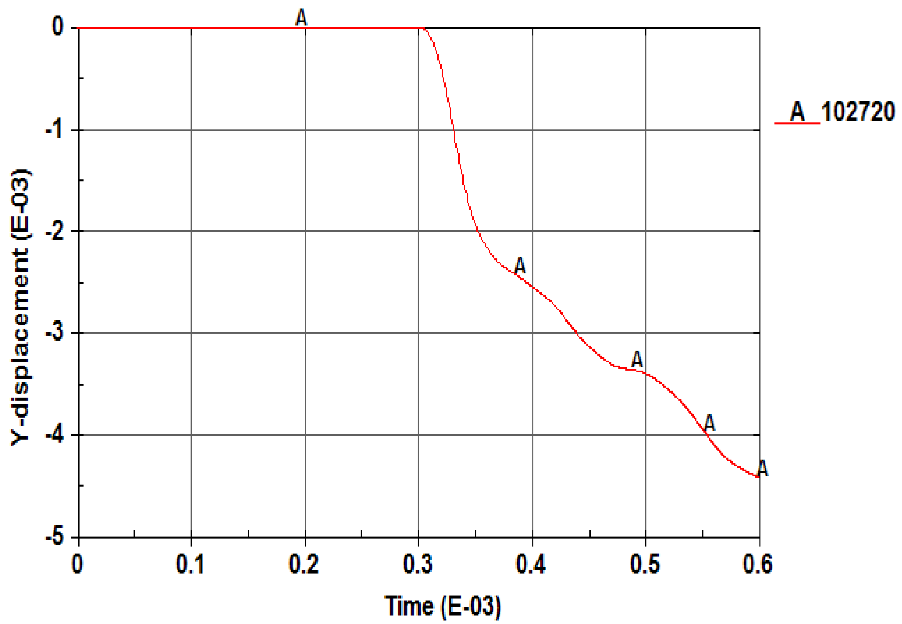
Lagrangian meshes in 3D

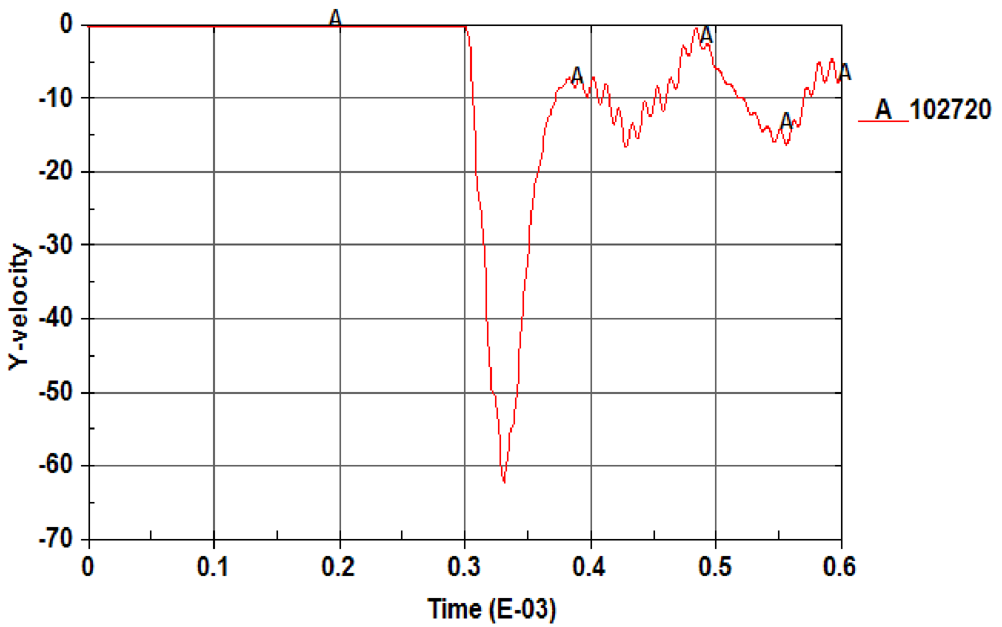



Time = 0
Contours of Effective Stress (v-m)
max IP. value
min=0, at node# 101965
max=0, at elem# 101965
Fringe Levels
0.000e+00
0.000e+00
0.000e+00
0.000e+00
0.000e+00
0.000e+00
Y
X

# Lagrangian Results

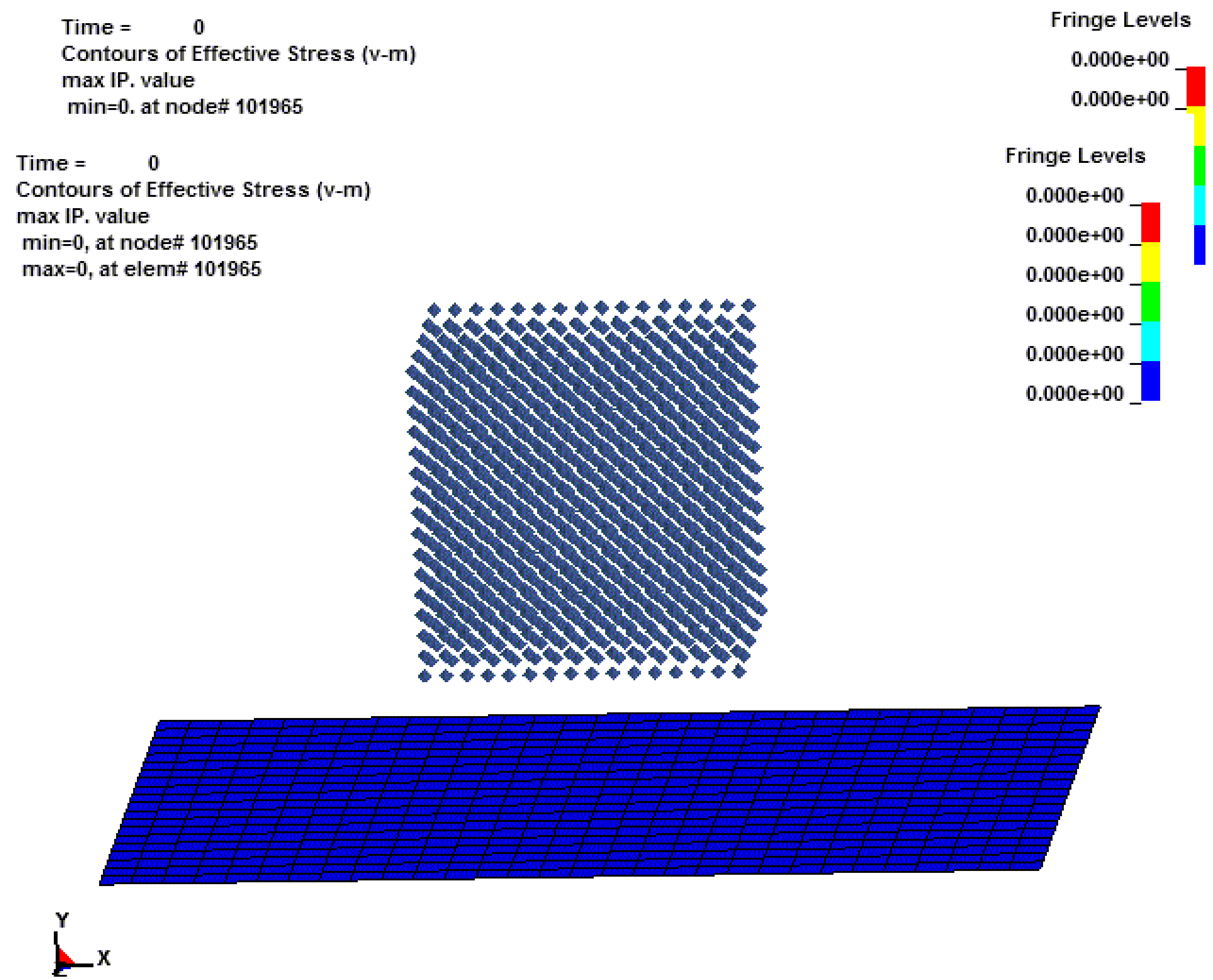


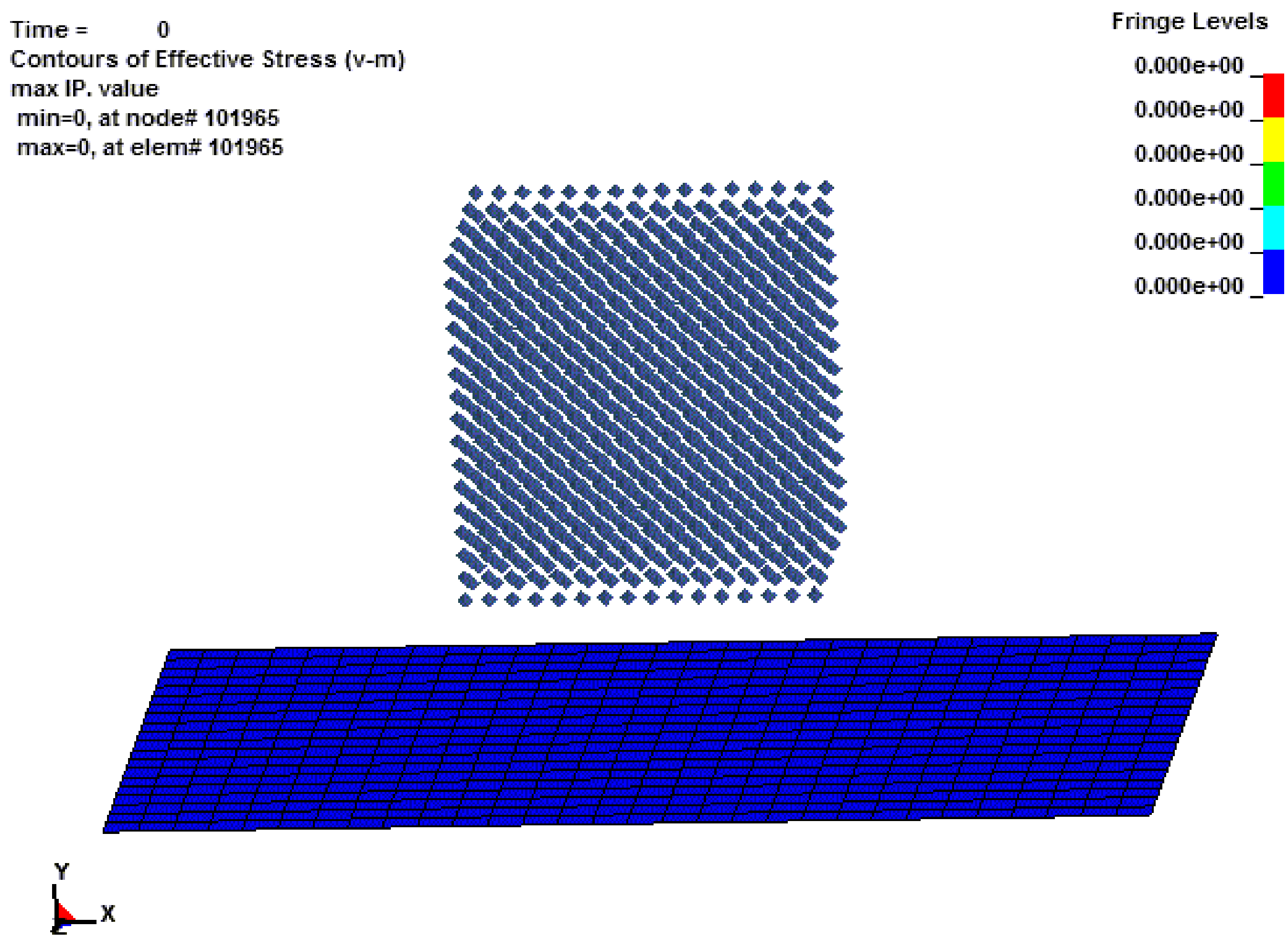



SPH meshes  in 3D

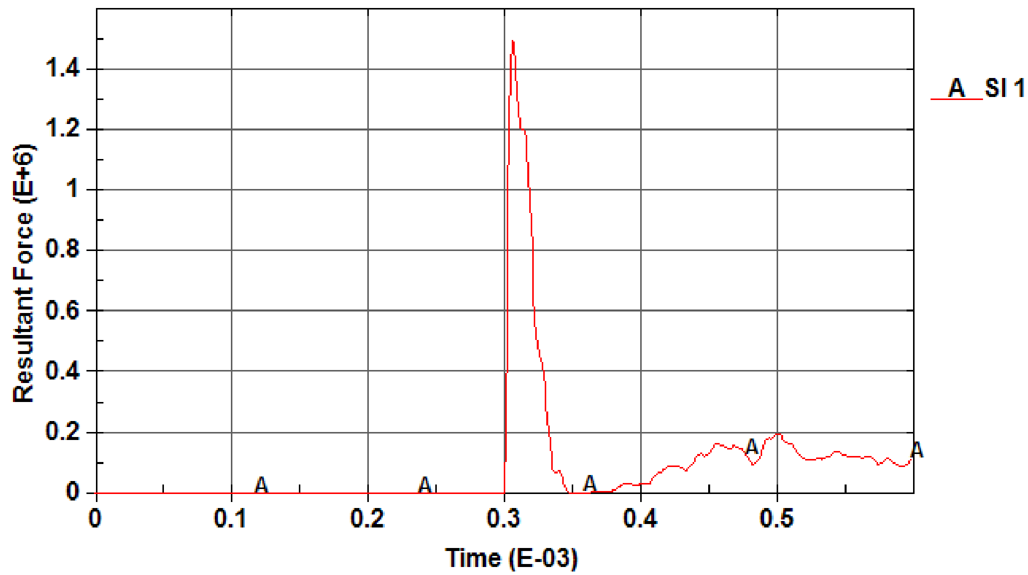
Same mesh for SPH and Lagrangian

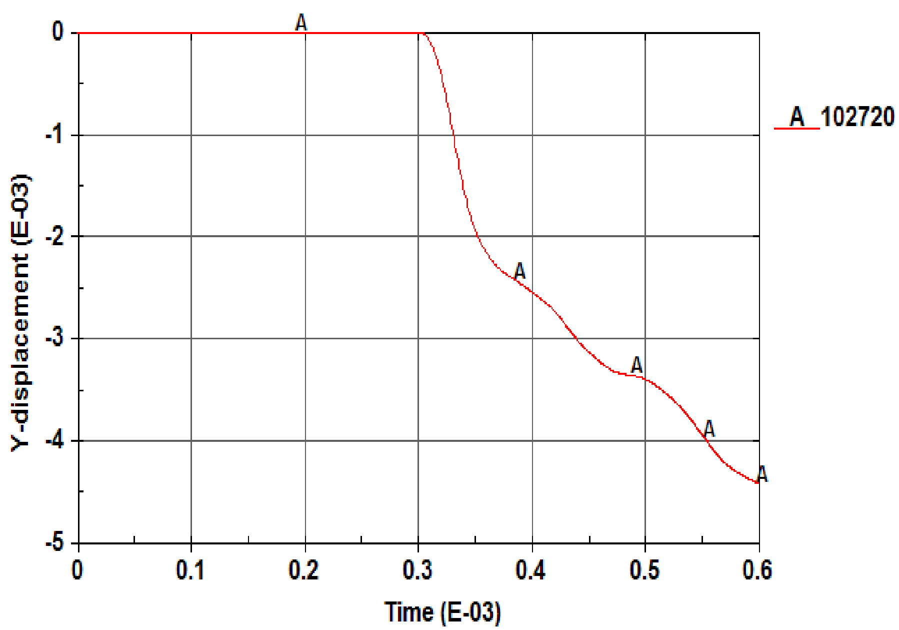


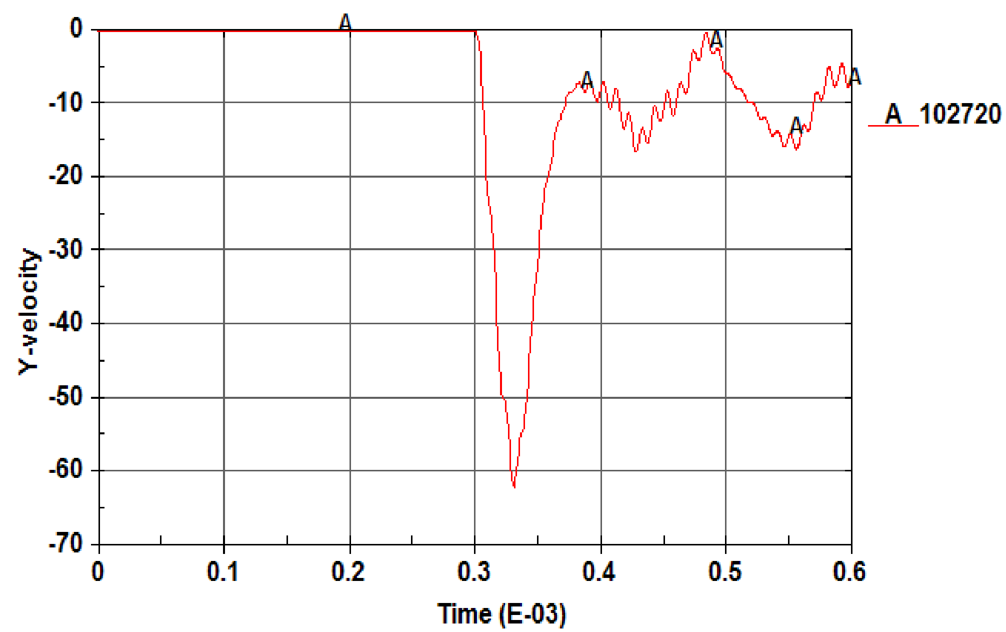
# SPH meshes in 3D



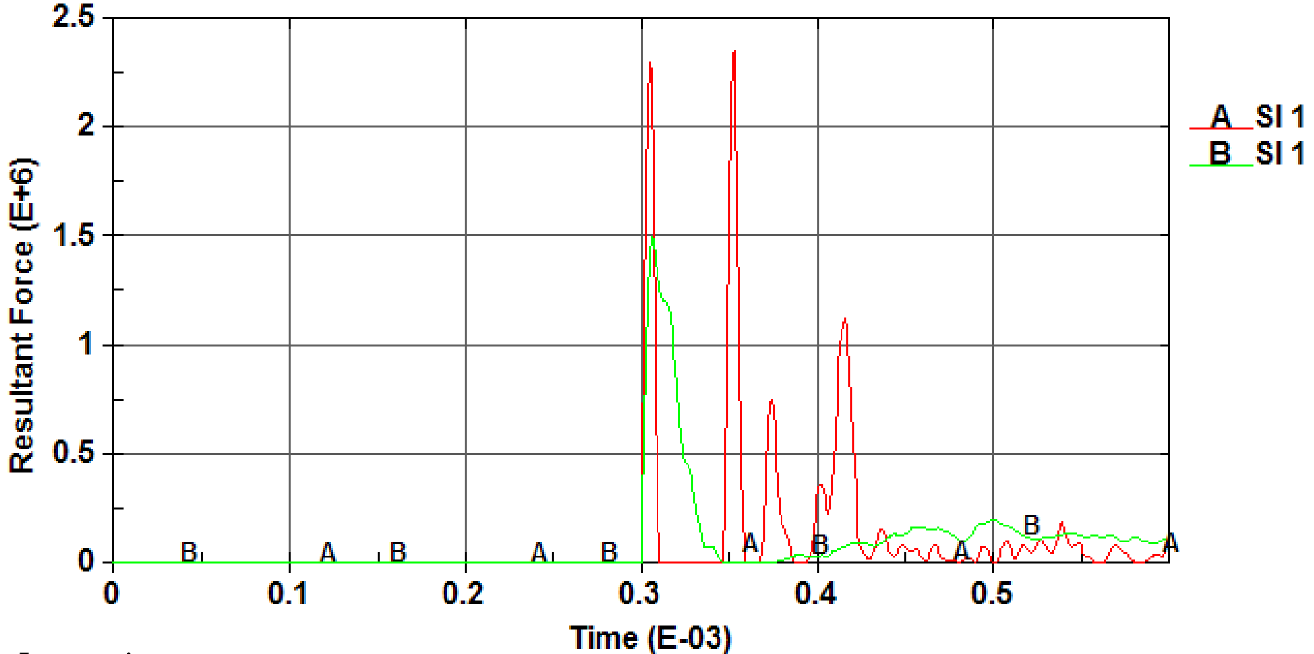
# Lagrangian and SPH meshes in 3D



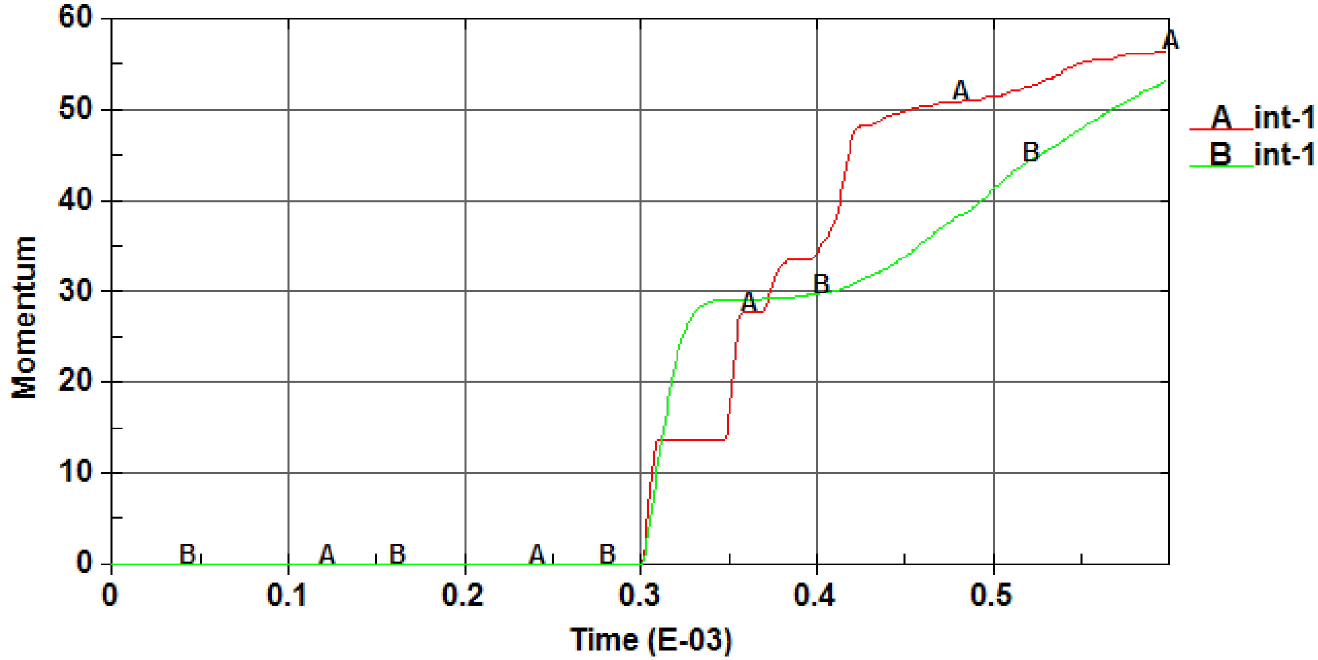



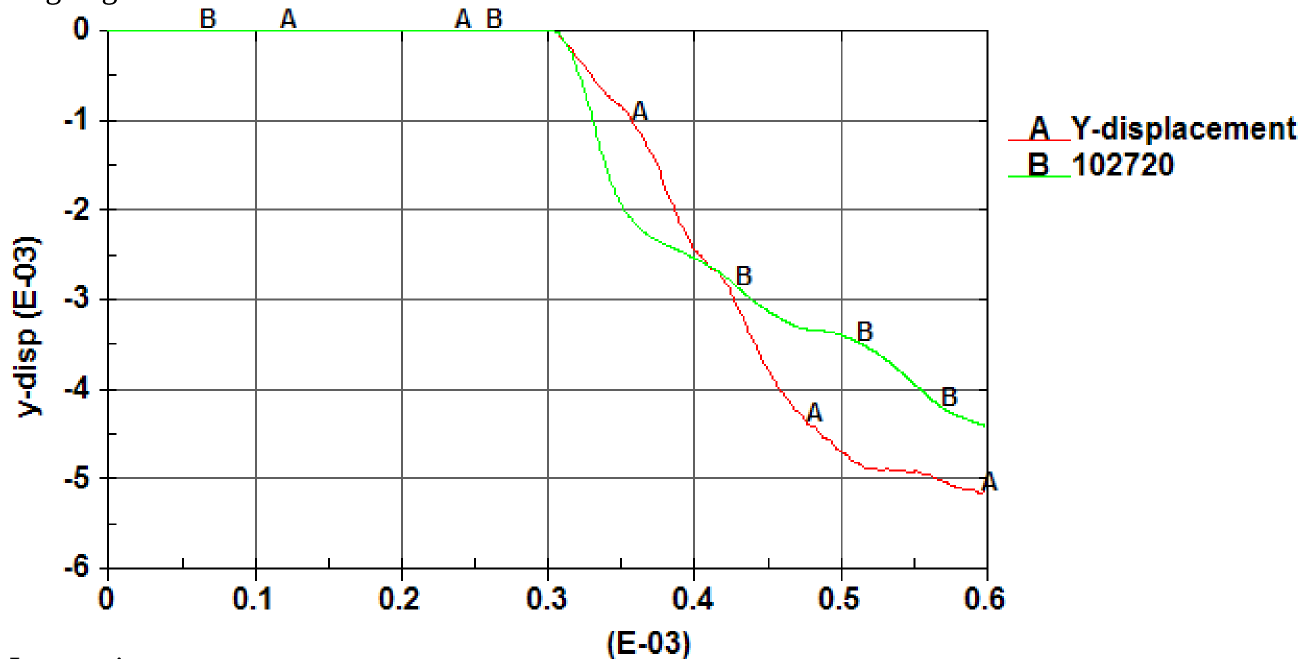
Resultant force:  A: SPH   B: Lagrangian



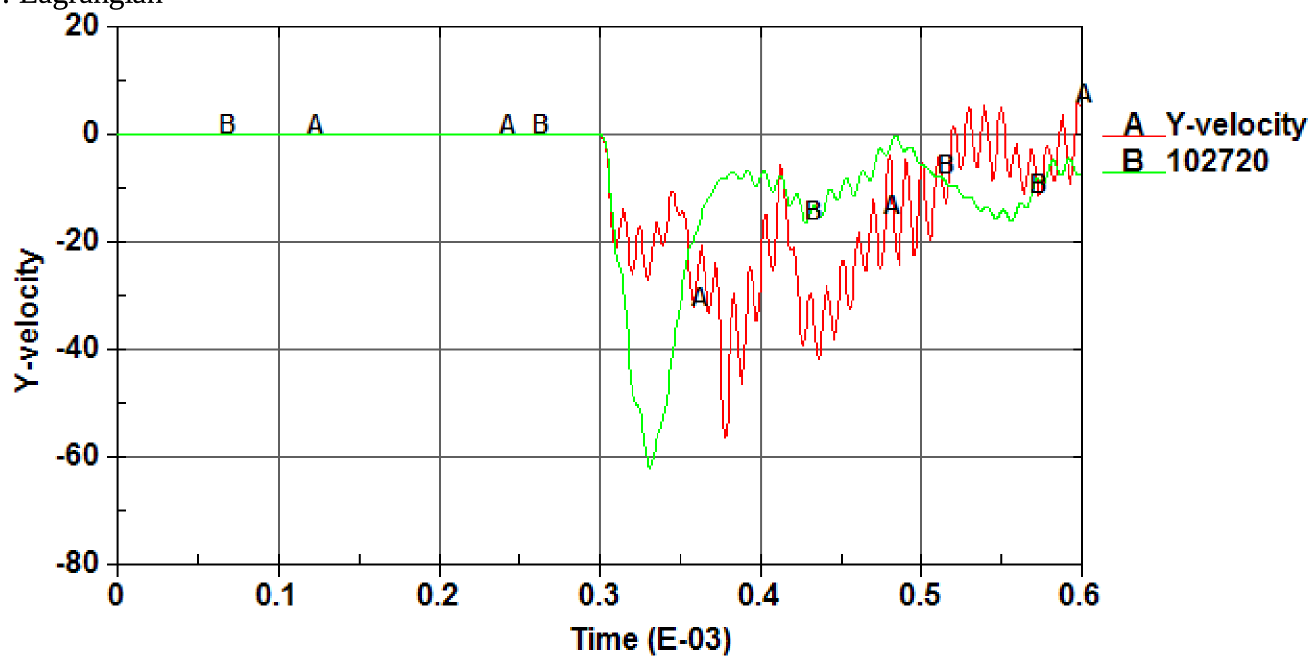

Momentum: A: SPH   B: Lagrangian :



Vertical disp A: SPH B: Lagrangian

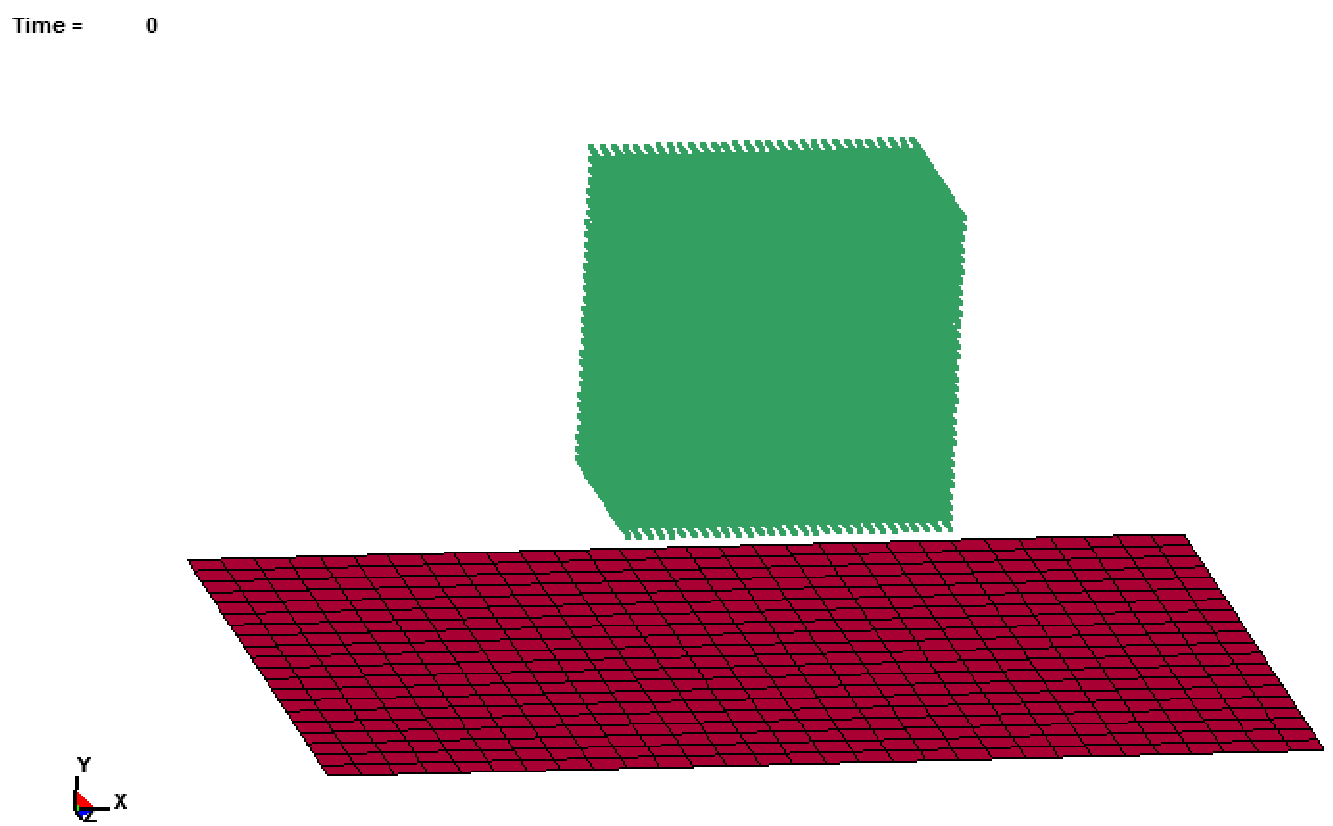


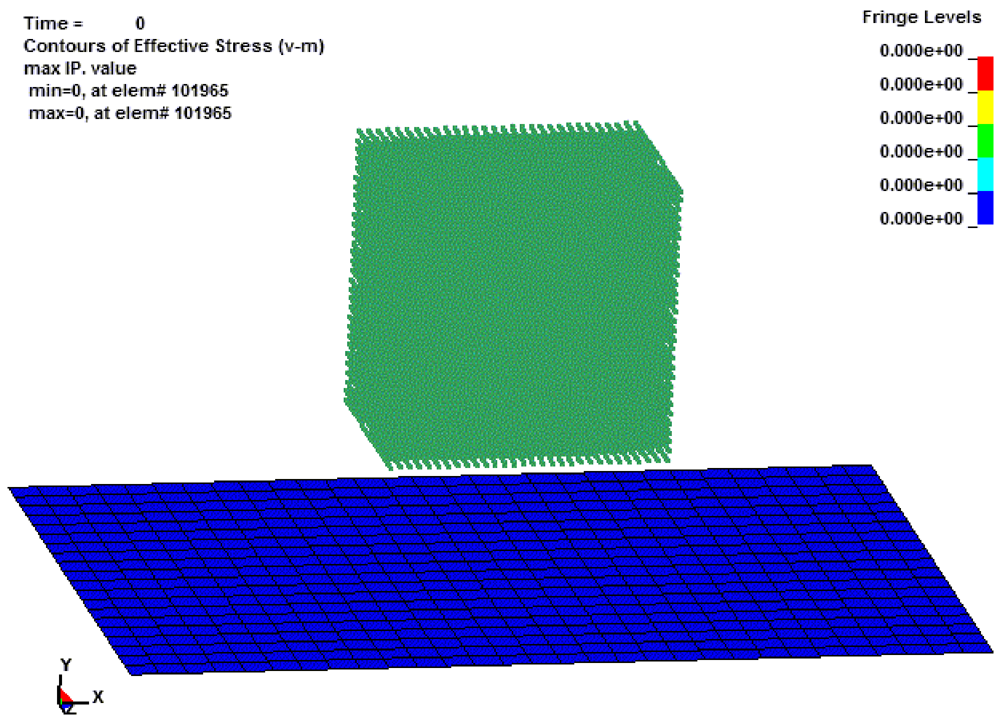

Momentum: A: SPH B: Lagrangian



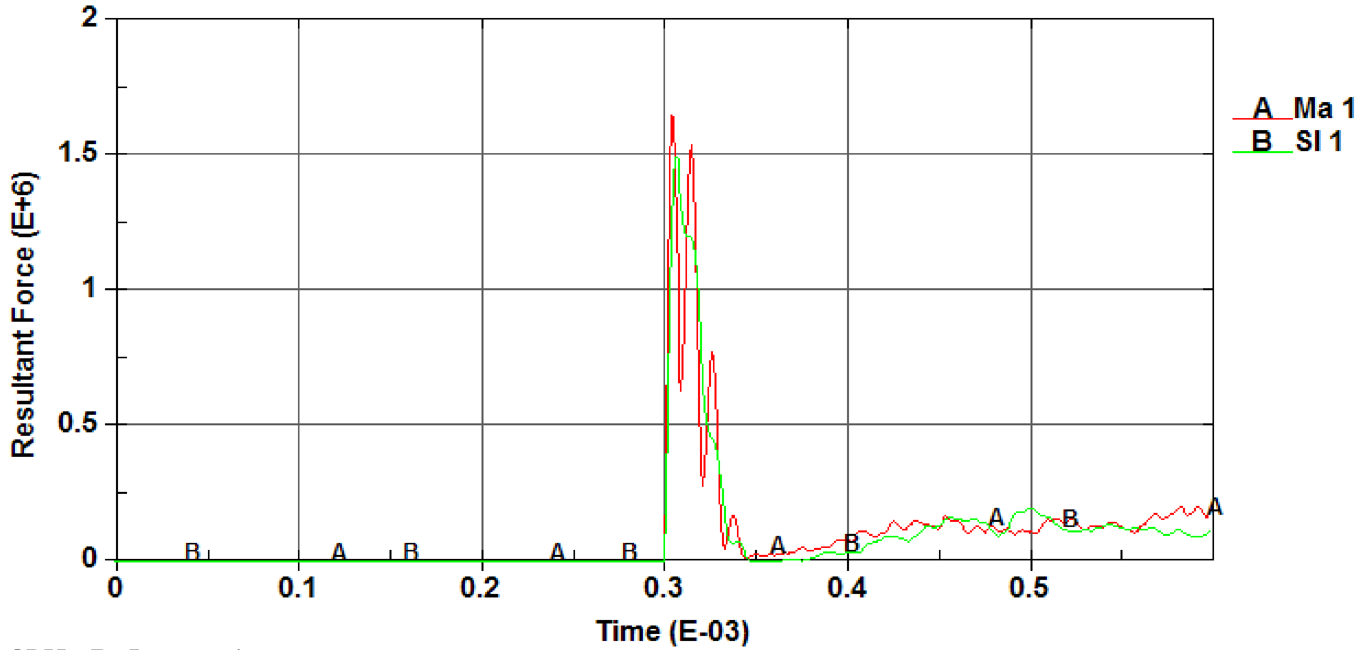
## Finer SPH meshes in 3D

SPH 3D mesh finer than Lagrangian

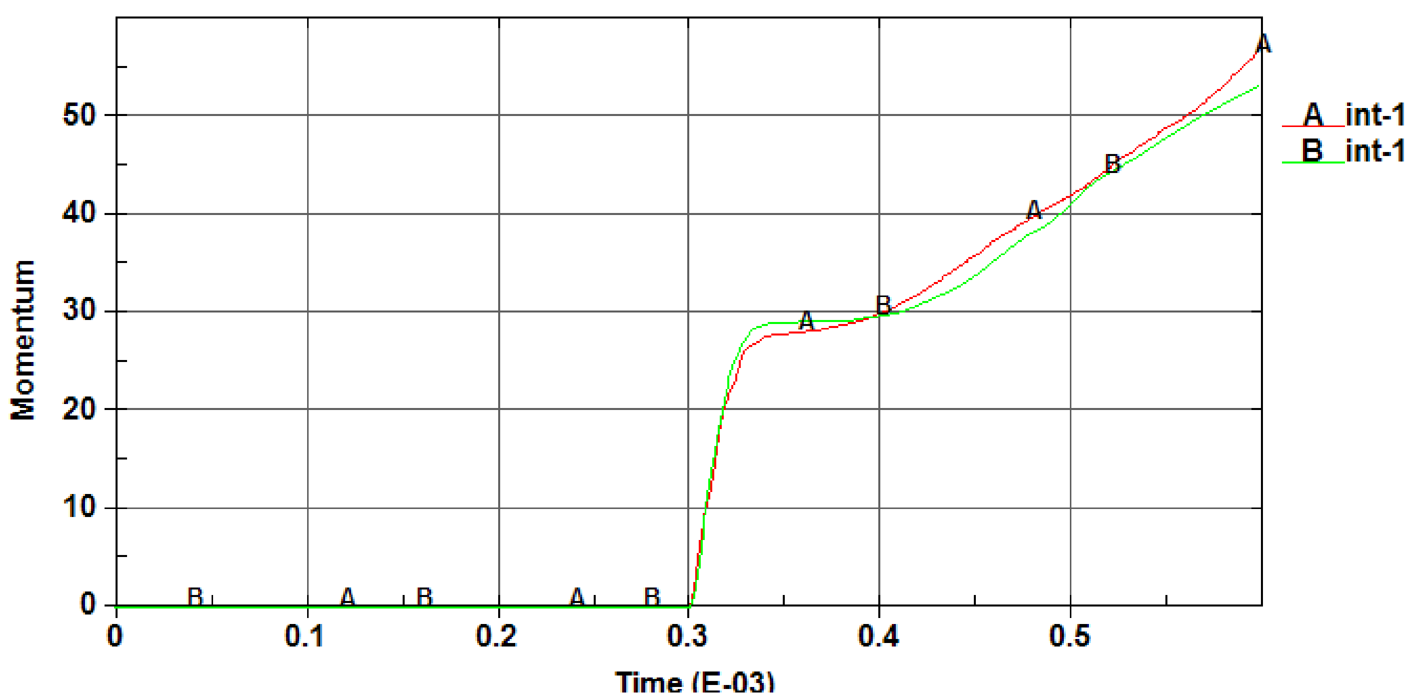



Time = 0
Contours of Effective Stress (v-m)
max IP. value
min=0, at elem# 101965
max=0, at elem# 101965
Fringe Levels
0.000e+00
0.000e+00
0.000e+00
0.000e+00
0.000e+00
0.000e+00
Y
X

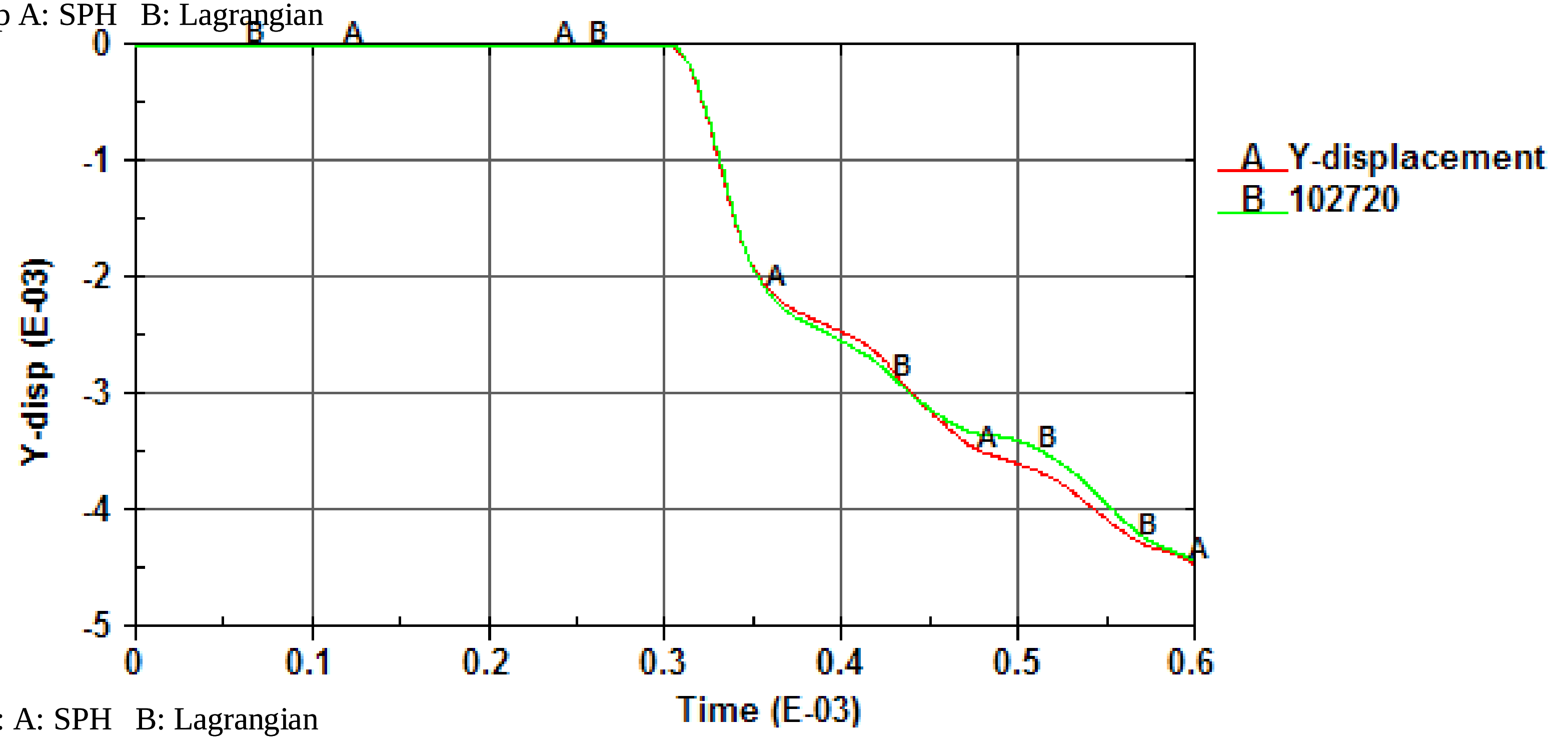
Resultant force: A: SPH B: Lagrangian



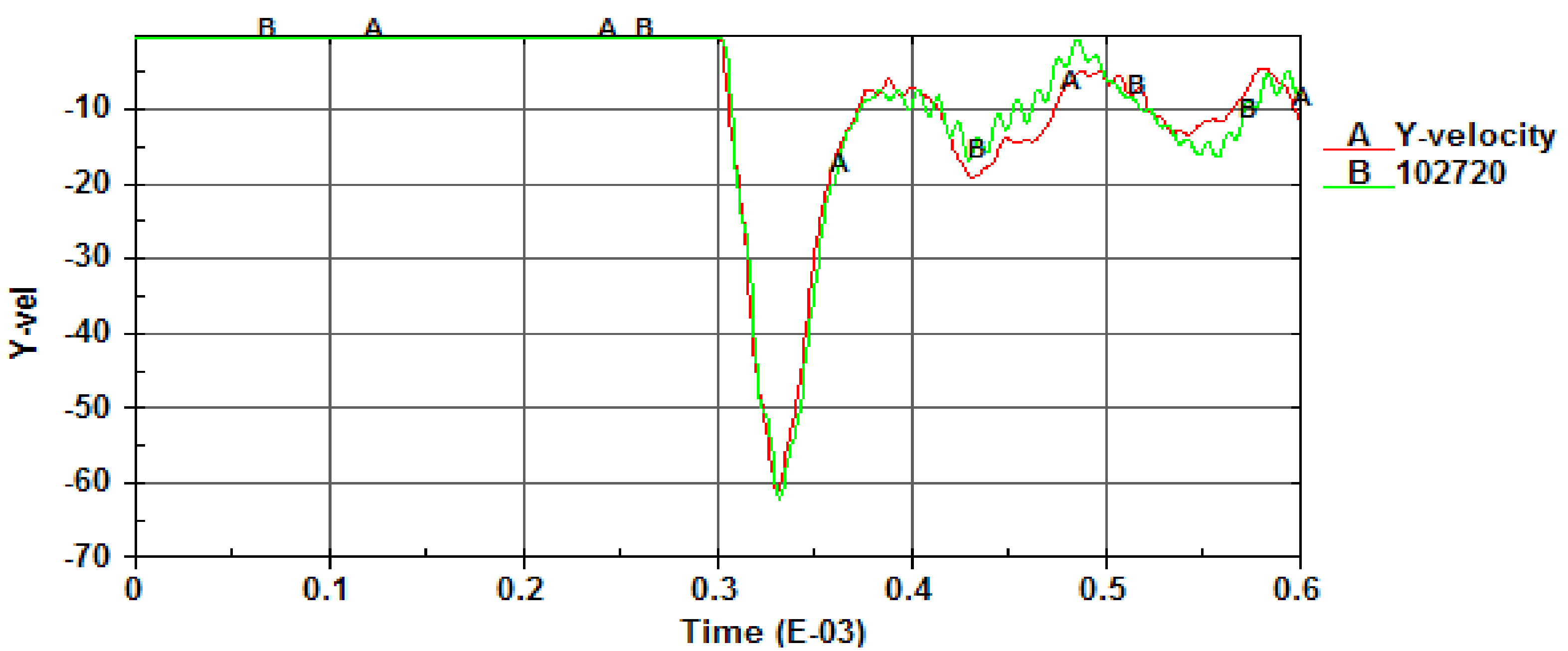

Momentum: A: SPH B: Lagrangian



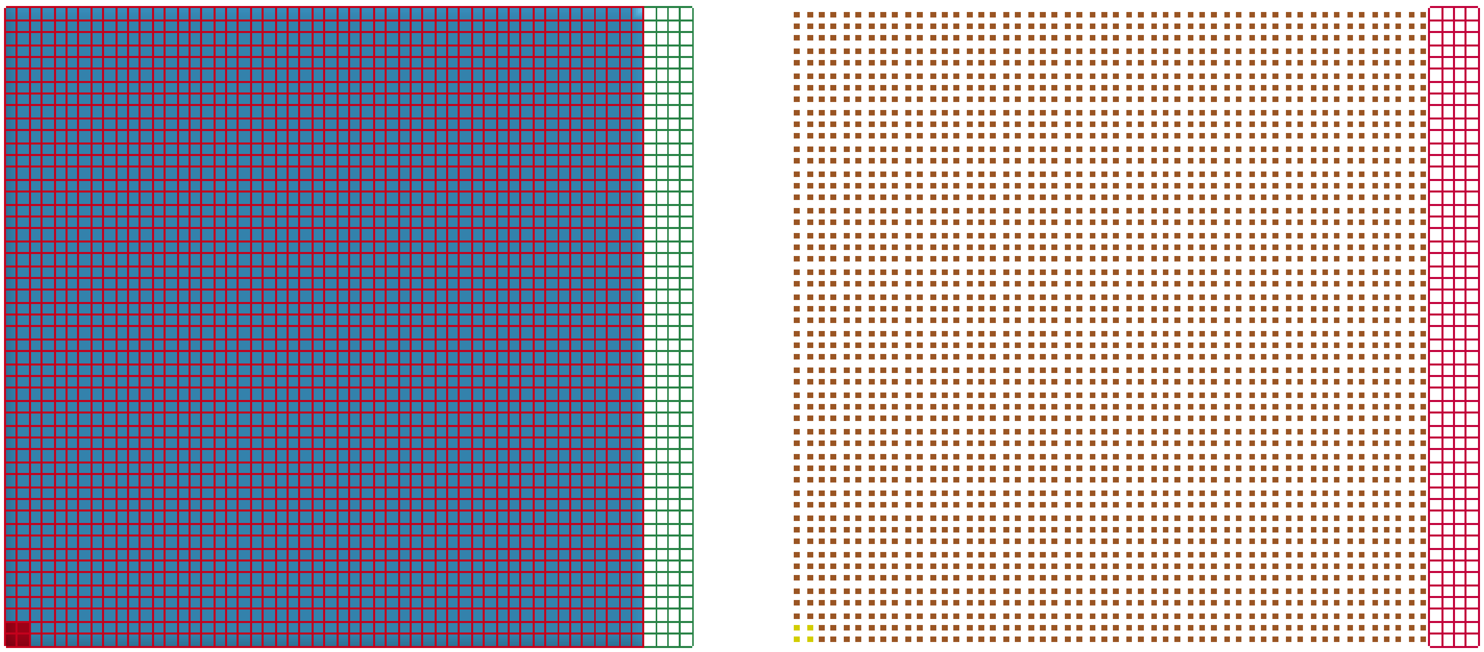
Vertical disp A: SPH   B: Lagrangian

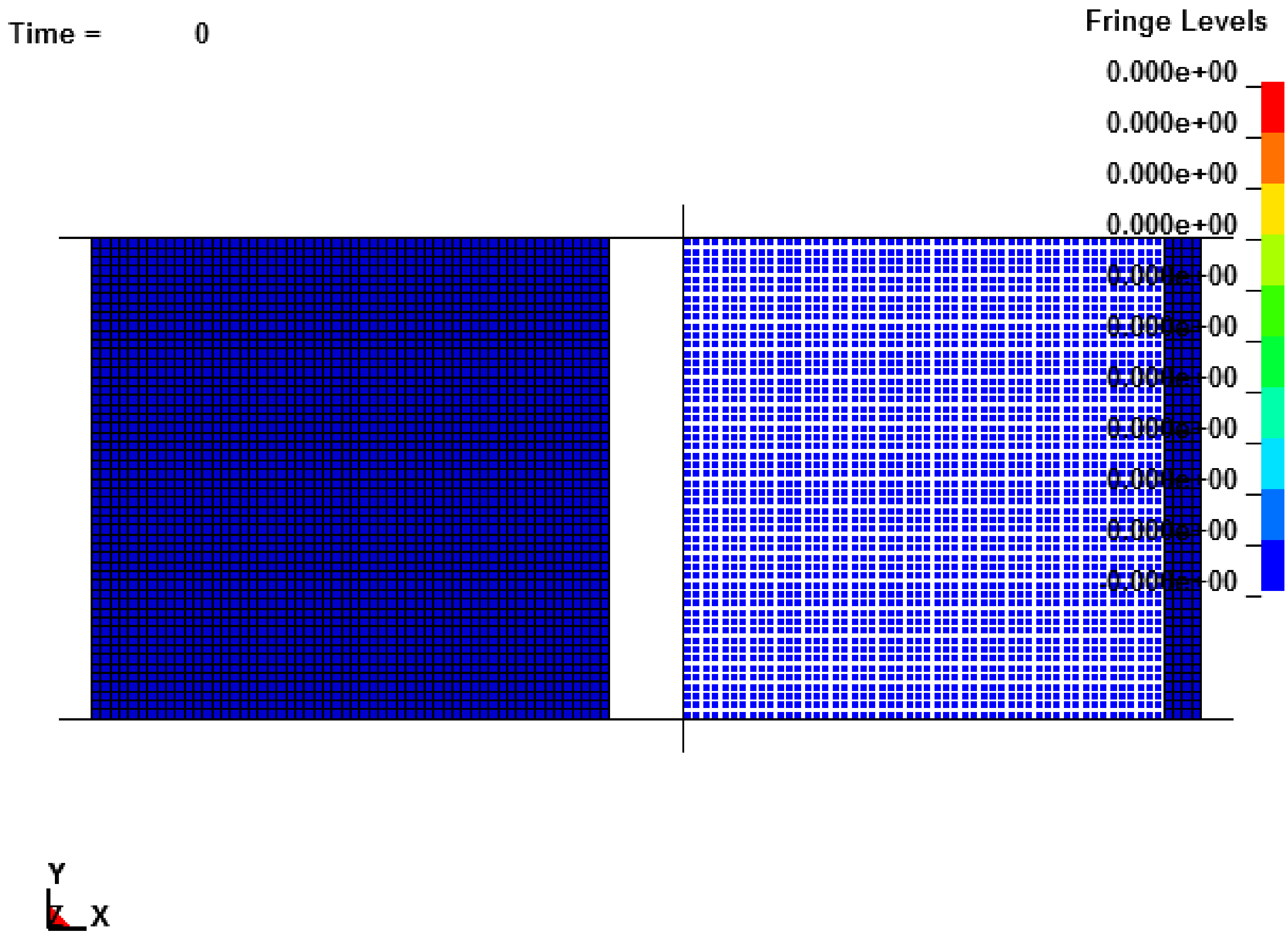



Vertical vel: A: SPH   B: Lagrangian



ALE and SPH for explosive problems





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Fringe Levels
0.000e+00
0.000e+00
0.000e+00
0.000e+00
0.000e+00
0.000e+00
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0.000e+00
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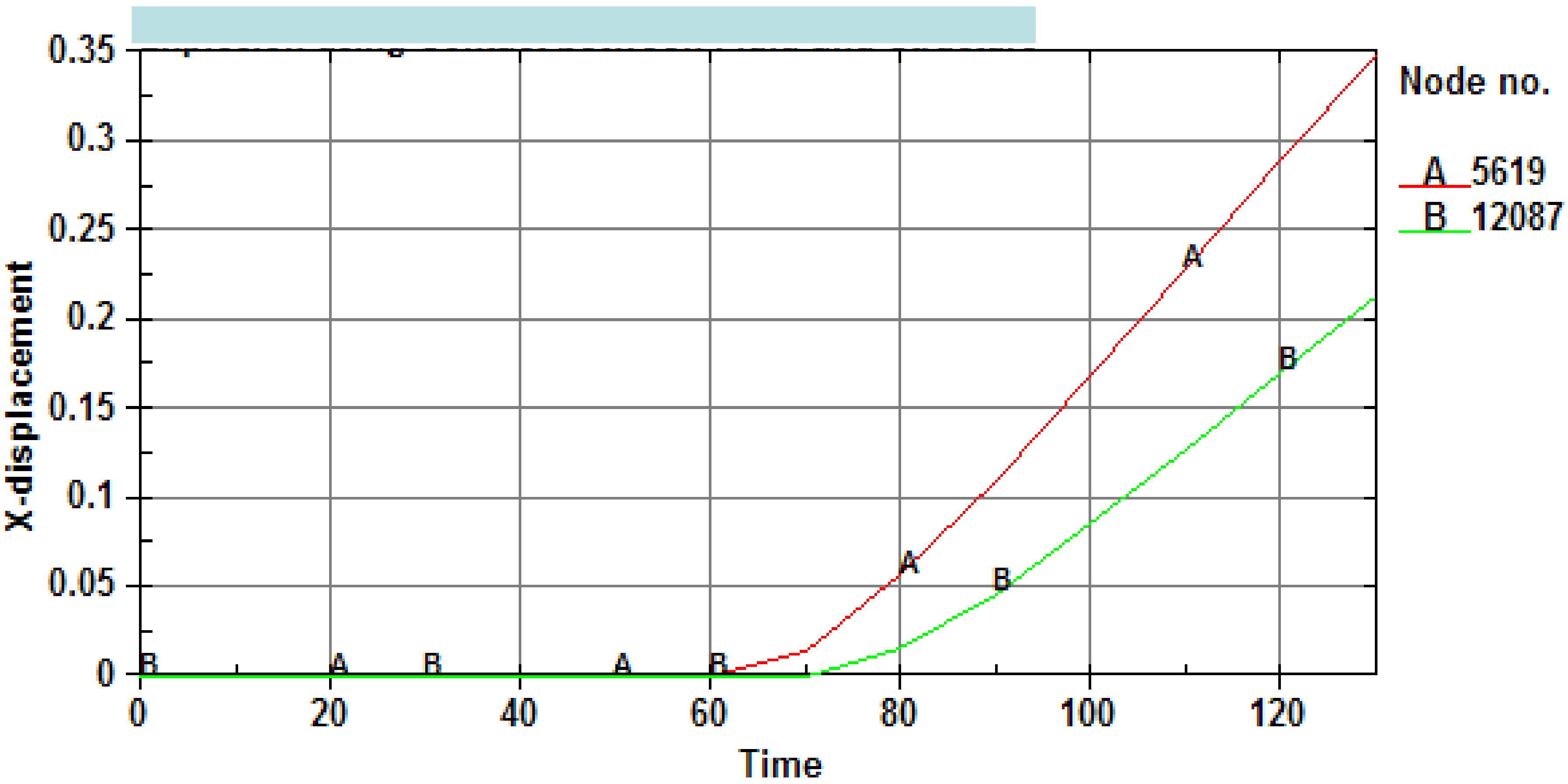
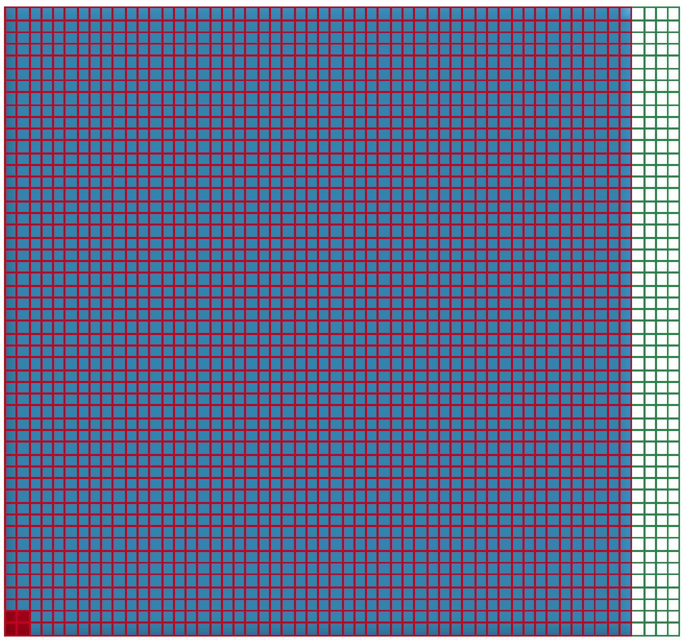
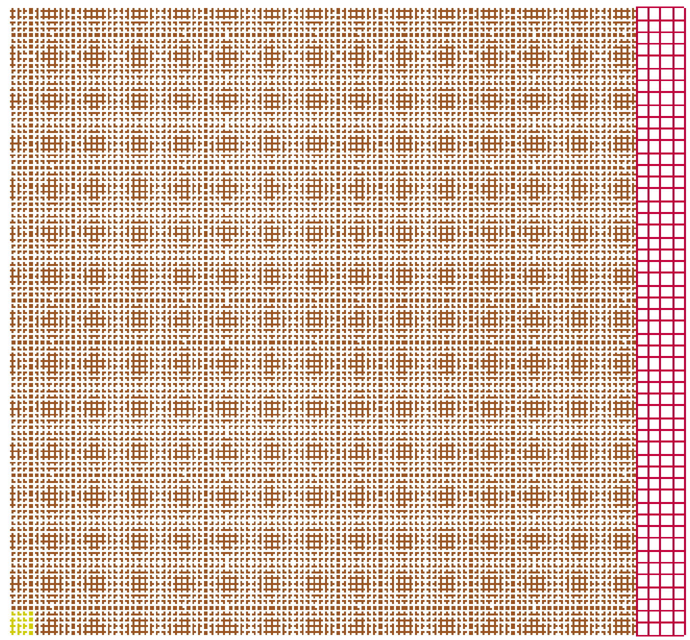
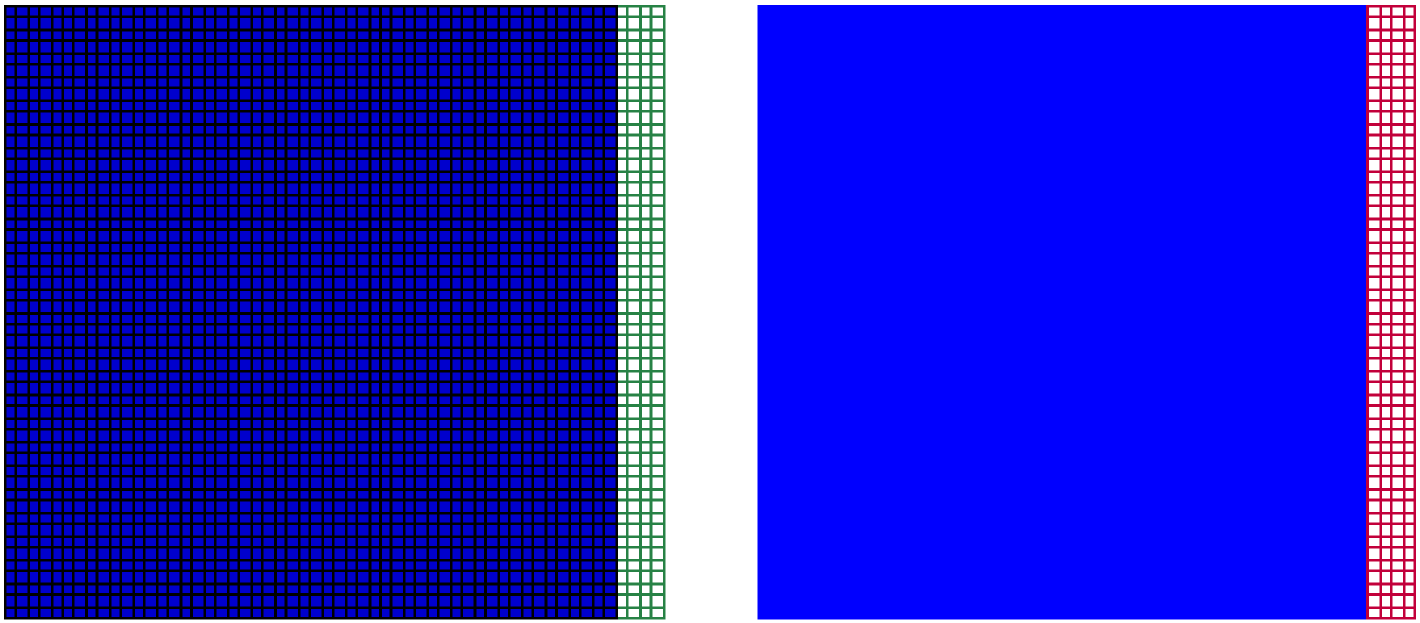
Y
Z
X

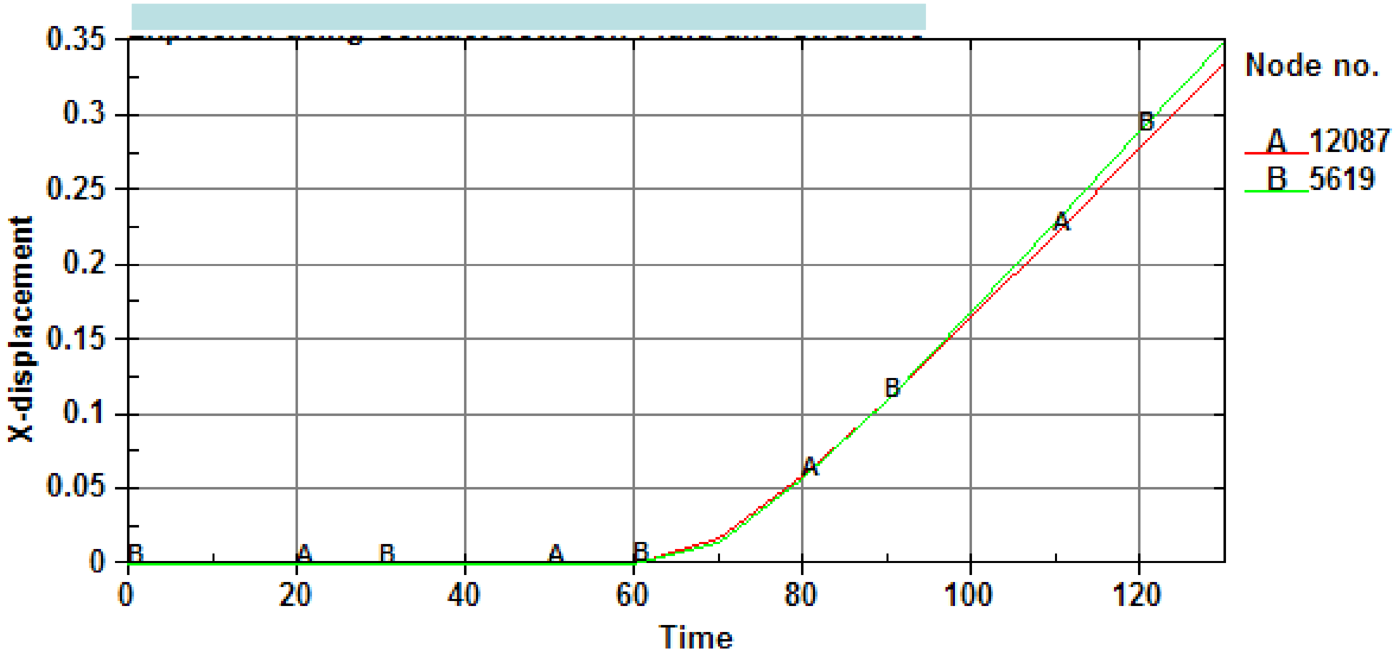

X-displacement
Time
Node no.
A 5619
B 12087
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
20
40
60
80
100
120
A
B

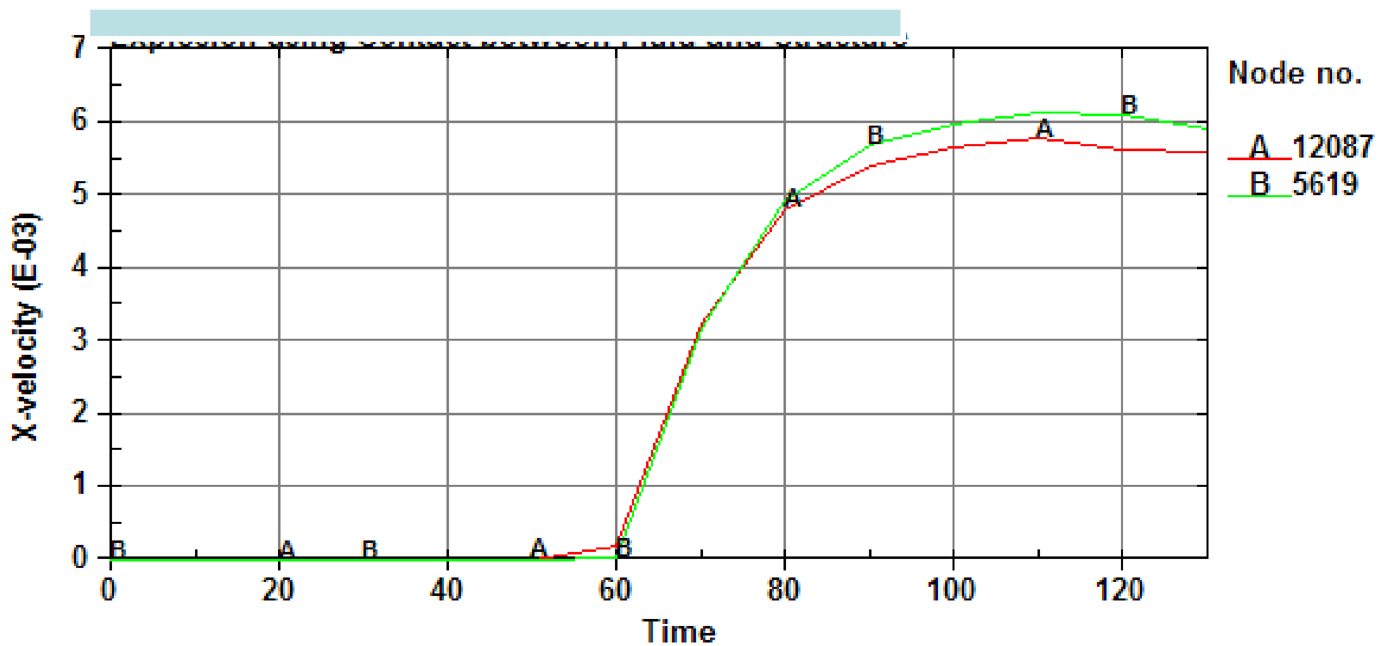




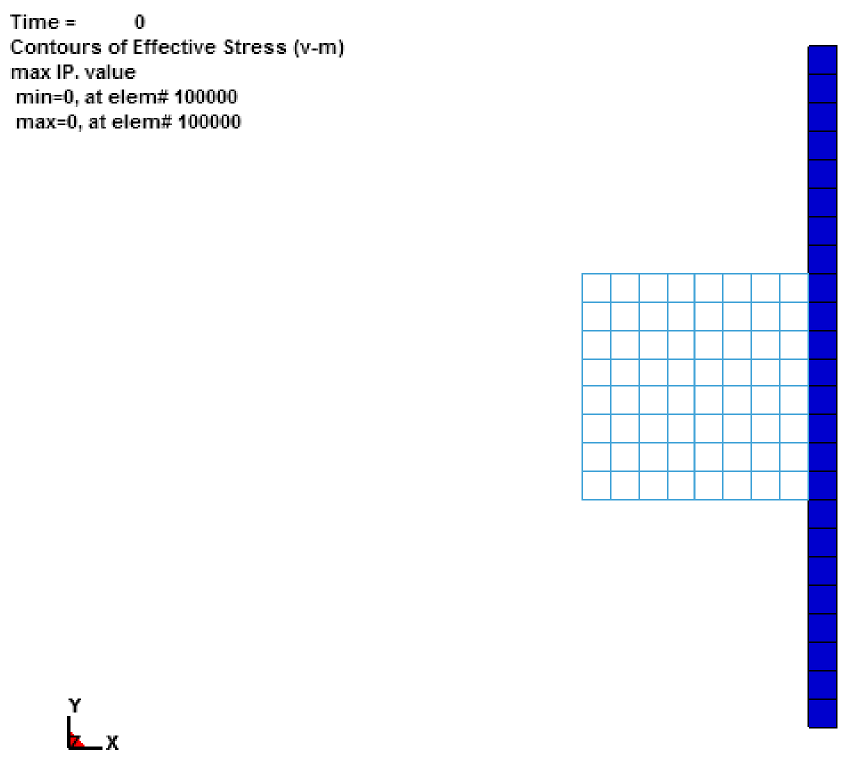
Time = 0




0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
X-displacement
0
20
40
60
80
100
120
Time
Node no.
A 12087
B 5619

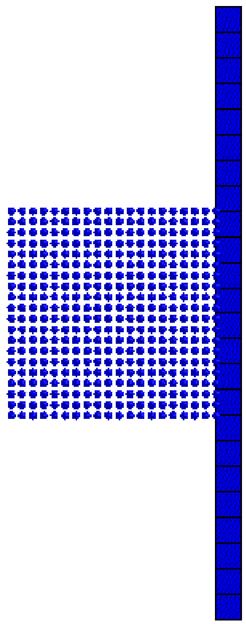

7
6
5
4
3
2
1
0
X-velocity (E-03)
0
20
40
60
80
100
120
Time
Node no.
A 12087
B 5619

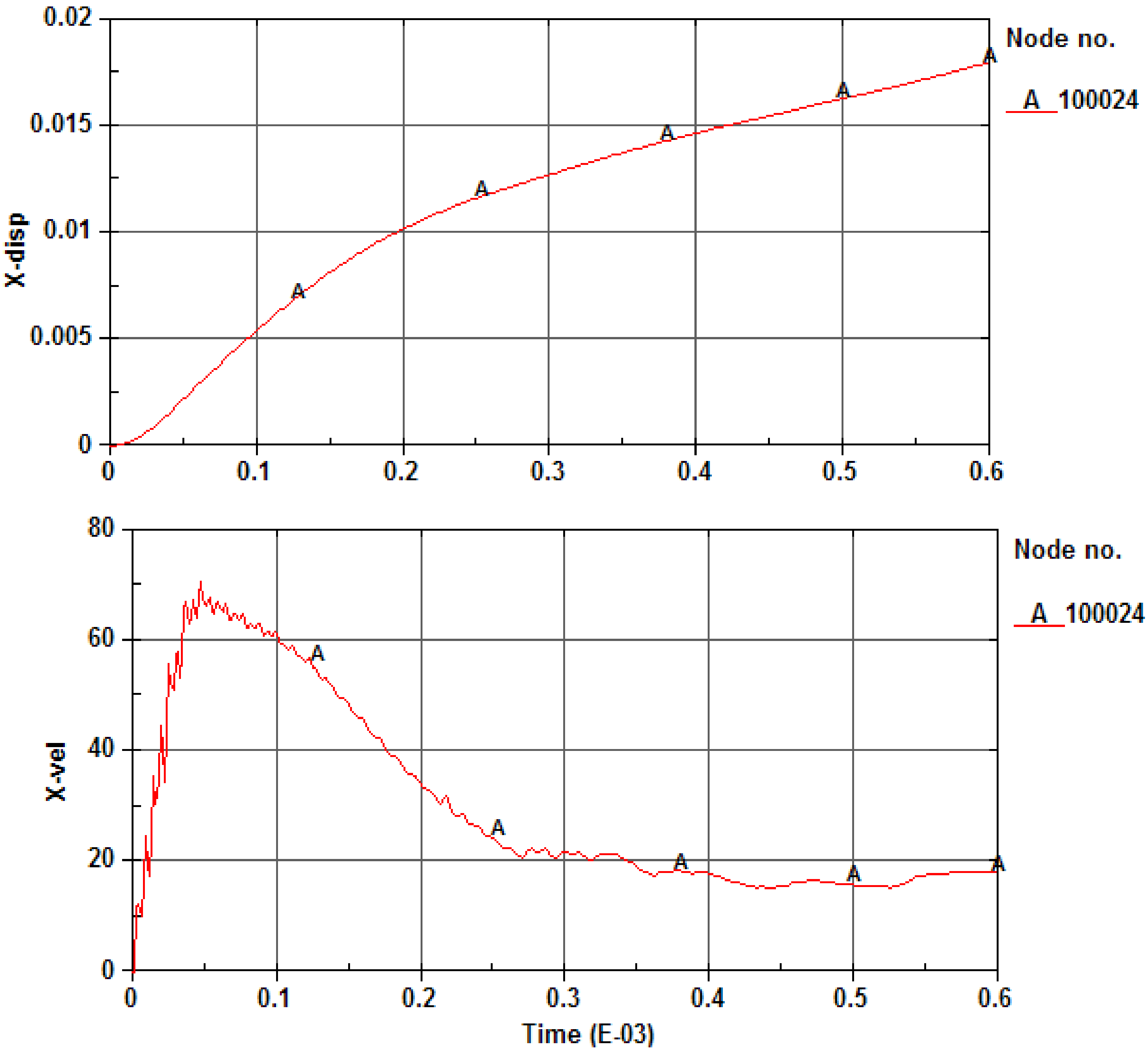
# 2D Lagrangian mesh



# Lagrangian and SPH meshes in 2D

2D Lagrangian mesh

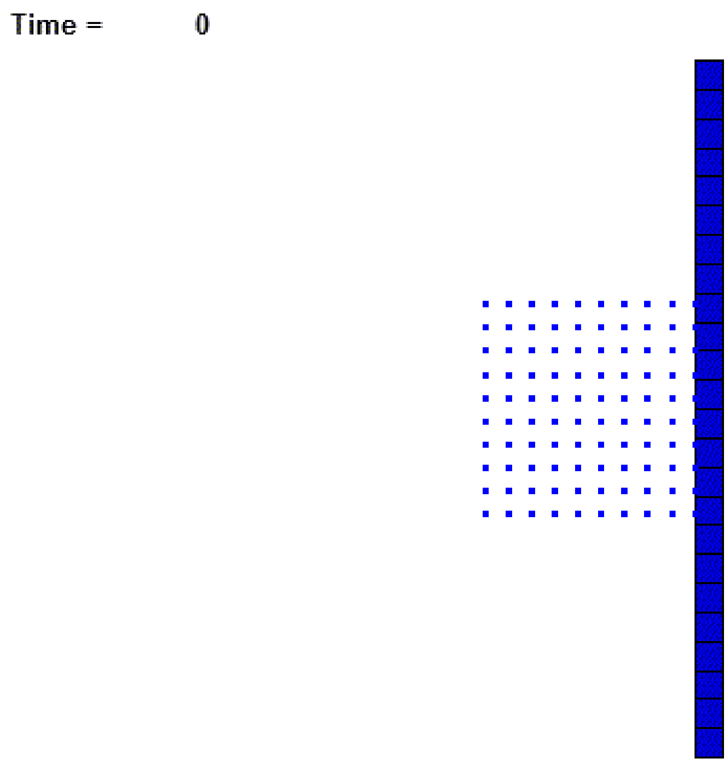




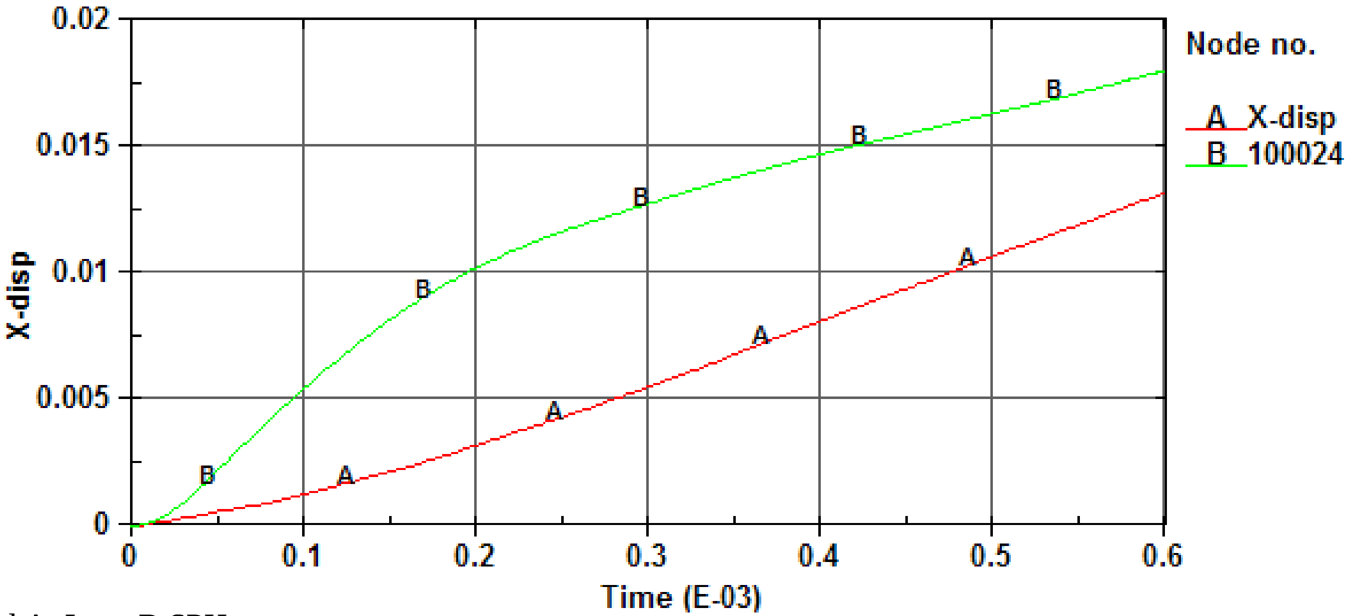
0.02
0.015
0.01
0.005
0
X-disp
0
0.1
0.2
0.3
0.4
0.5
0.6
Node no.
A 100024
80
60
40
20
0
X-vel
0
0.1
0.2
0.3
0.4
0.5
0.6
Time (E-03)
Node no.
A 100024

# Lagrangian and SPH meshes in 2D

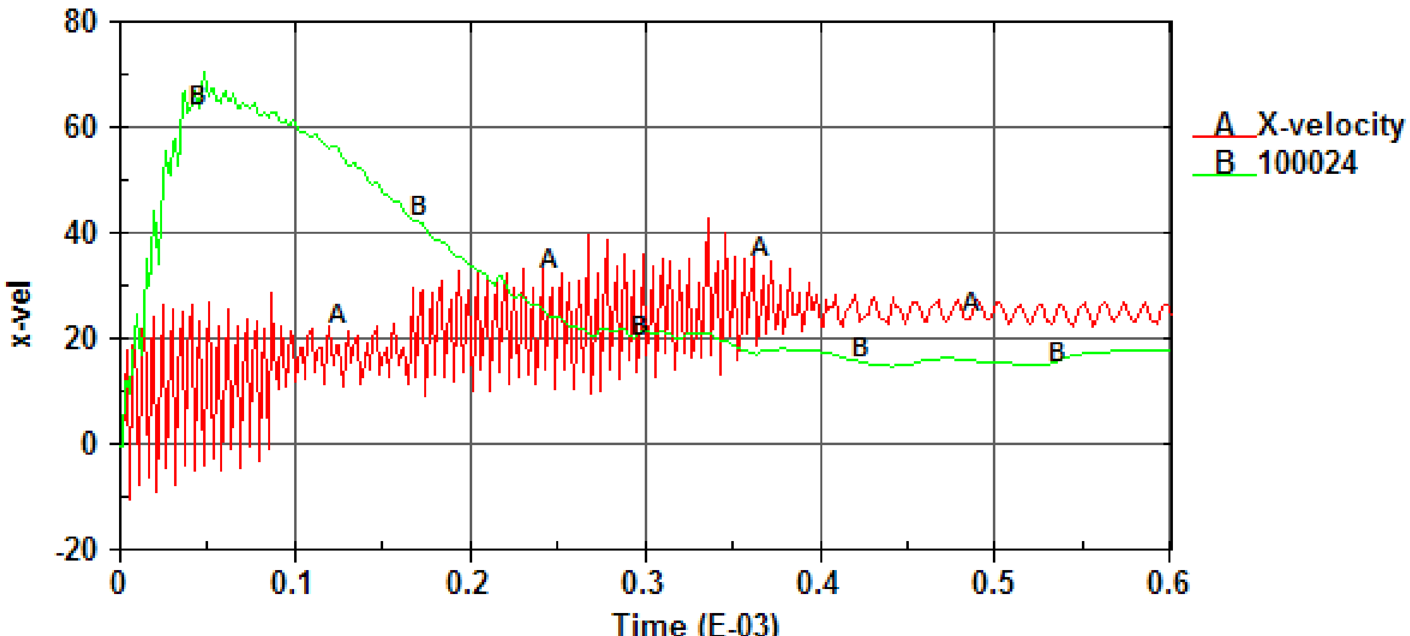
2D SPH mesh

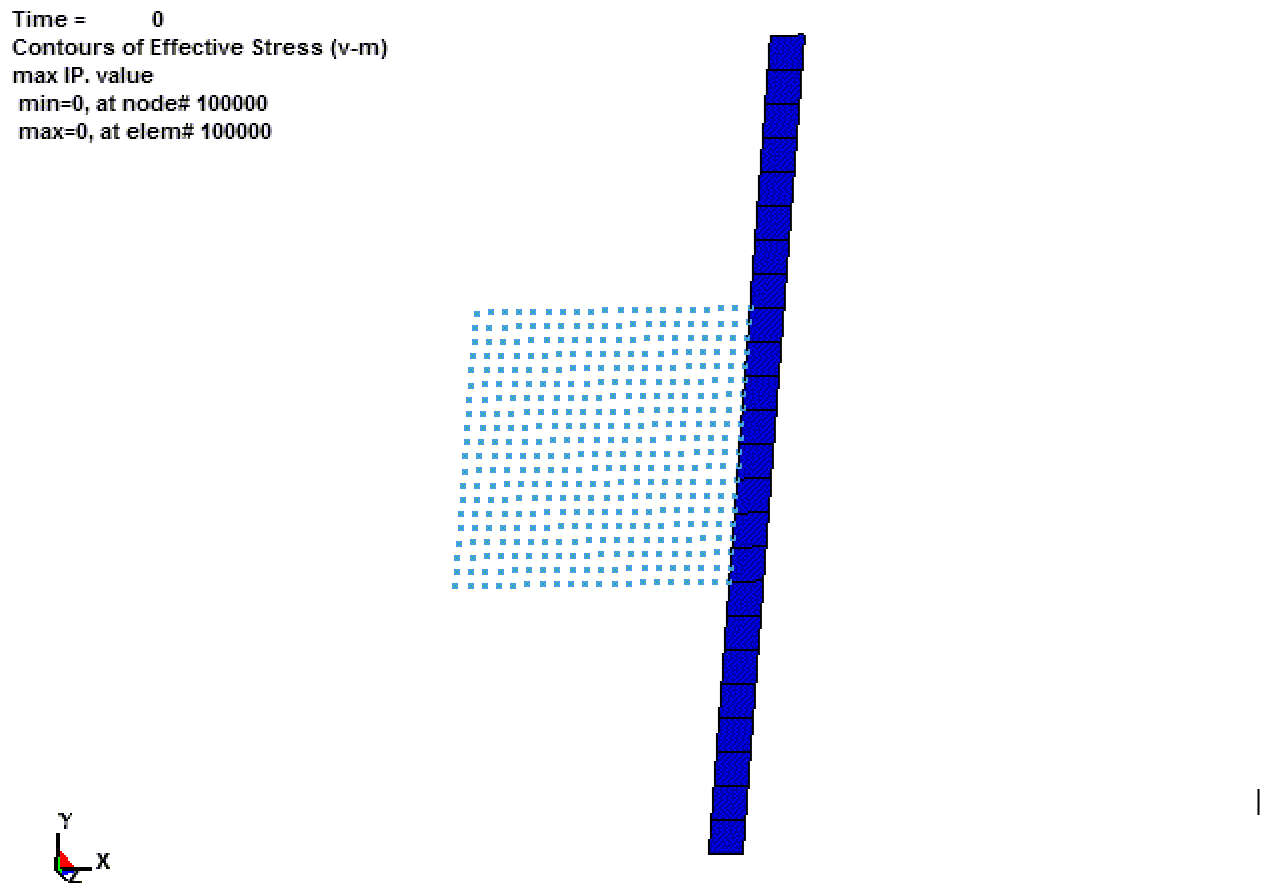


xdisp A: Lag   B: SPH

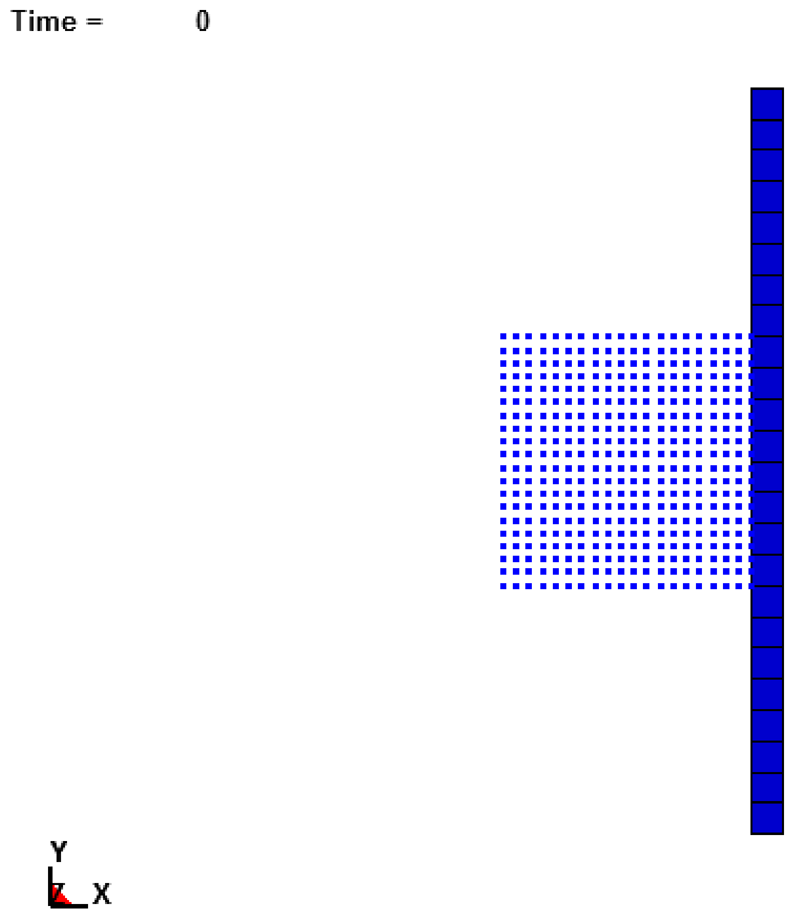



xvel A: Lag   B:SPH

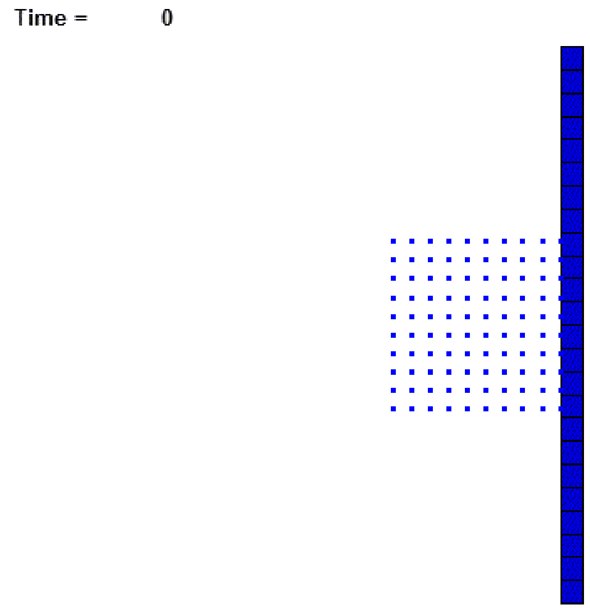


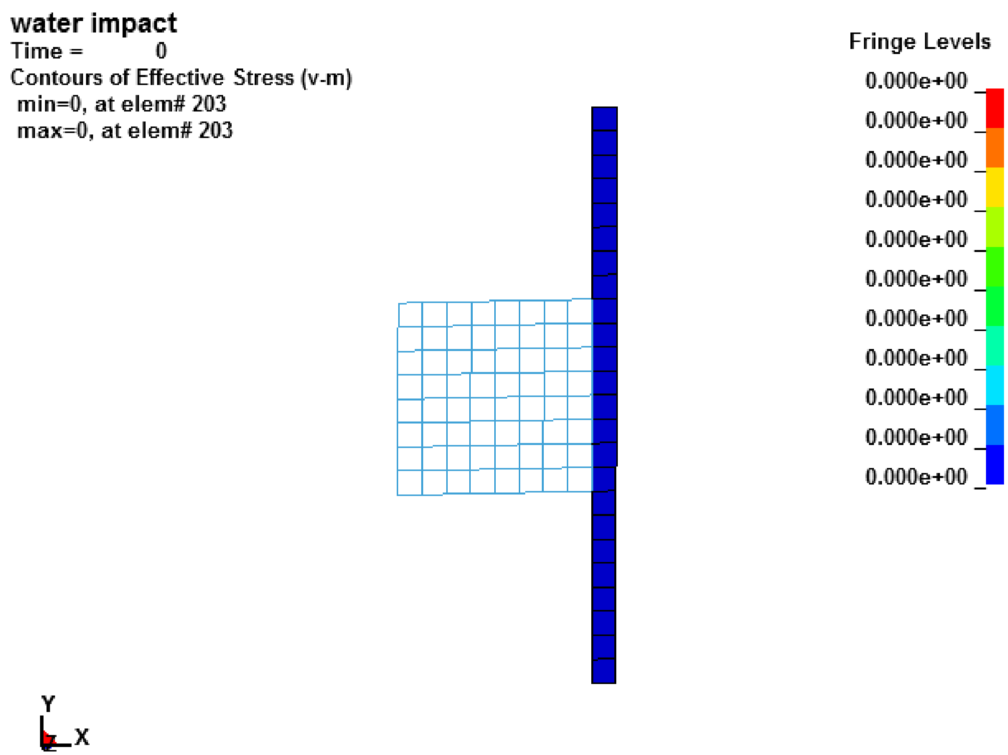

Time = 0
Contours of Effective Stress (v-m)
max IP. value
min=0, at node# 100000
max=0, at elem# 100000
Y
X
Z

2D finer SPH

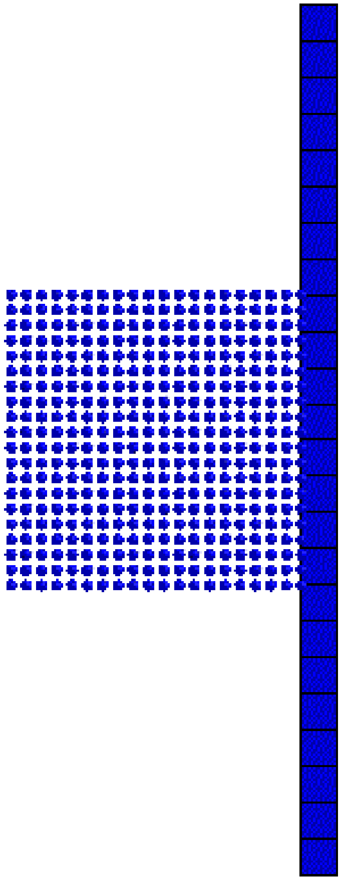


2D SPH  and Lag mesh

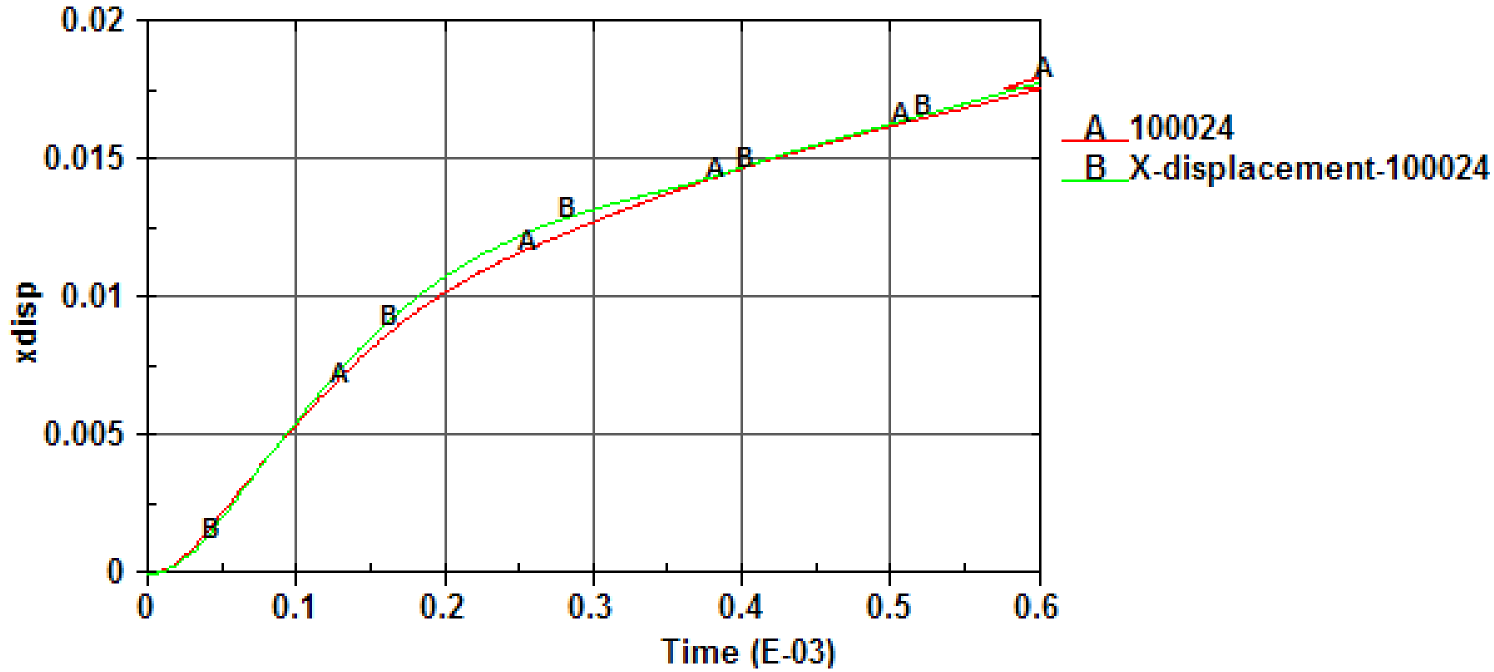


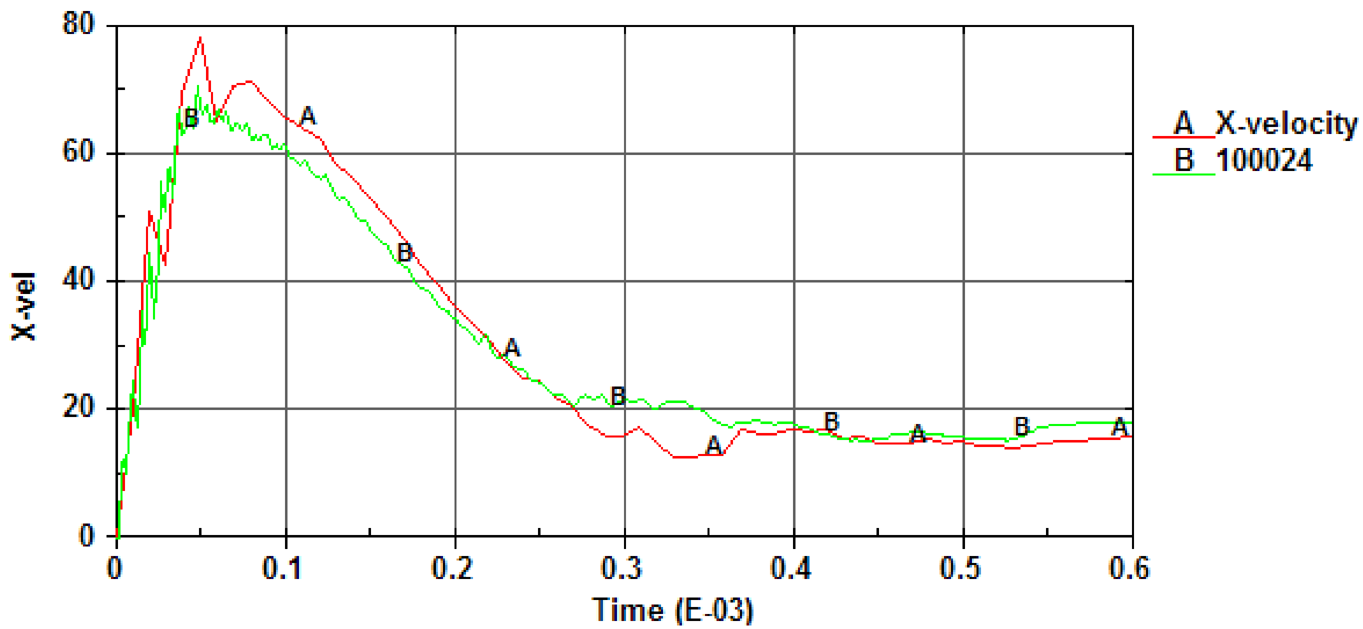
# 2D fine and coarse SPH



xdisp A: Lag   B: SPH

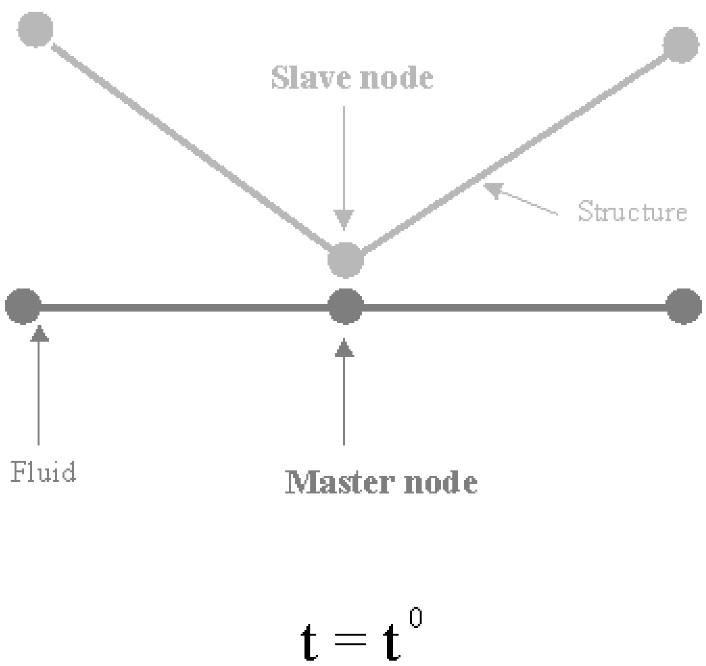


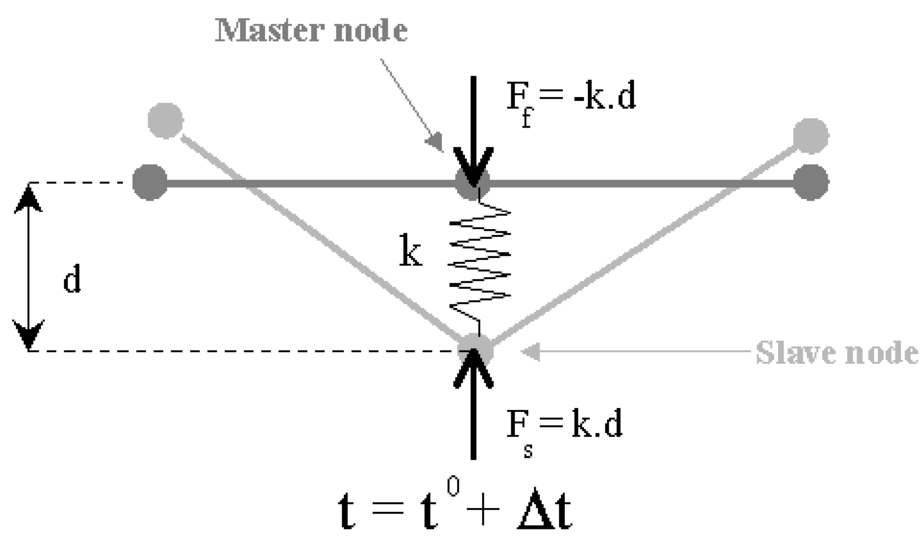

Xvel:  A: Lag   B: SPH



# Explicit **Contact Algorithm**

## 2) Penalty Based Contact.

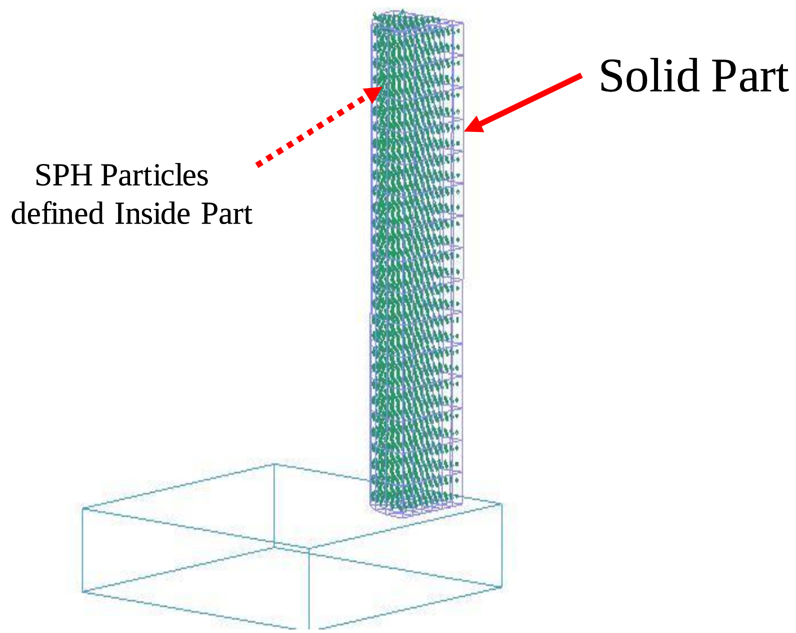


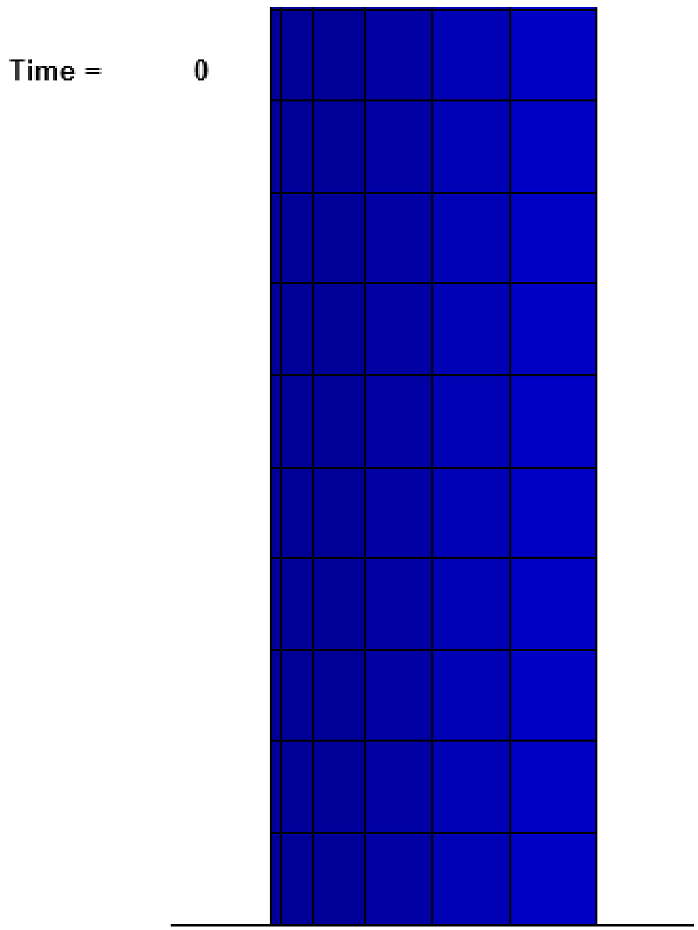
## SPH Adaptive mesh

After element erosion, we loose element mass and momentum

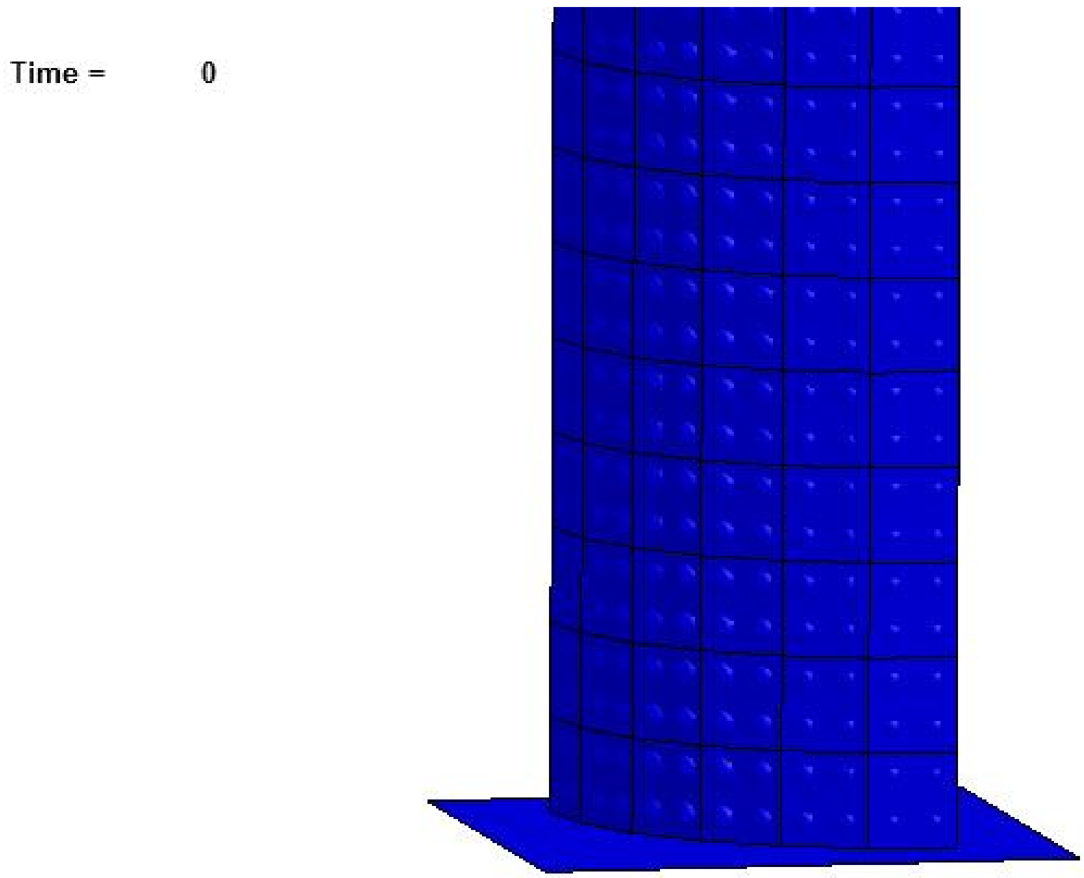
To keep mass and momentum of eroded element, the eroded element is replace by One or more SPH particles



Eroded element are not replaced by particles



Eroded element replaced by particles



Thank You